Free Vibration Analysis Technique of Large Structures

Man Cheol- Kim, In Won- Lee*

Ph.D. Student, Department of Civil Engineering, KAIST

*Professor , Department of Civil Engineering, KAIST

ABSTRACT

An efficient numerical method which can calculate the natural frequencies and mode shapes for large structural systems is presented. This method applies the accelerated Newton-Raphson method to eigenproblems. If eigenvalues are not multiple, this method can calculate the natural frequencies and mode shapes without any numerical instability which may be encounterd in the inverse iteration method with shift. The efficiency of this method is verified by comparing convergence and solution time for numerical examples with those of the subspace iteration method and the determinant search method.

1. INTRODUCTION

The analysis of a number of physical problems requires the solution of an eigenproblem. It is therefore natural that with the increased use of computational methods operating on discrete representations of physical problems, the development of efficient techniques for the calculation of eigenvalues and eigenvectors has attracted much attention. In particular, the use of finite element technique can lead to large systems of equations, and the efficiency of an overall response analysis can depend to a significant degree on the effectiveness of the solution of the required eigenvalues and eigenvectors.

The determinant search method[1,2] and the subspace iteration method[1,3,5] have been mainly used for solving eigenproblems. The determinant search method consists of the polynomial iteration method, the Sturm sequence property and the vector iteration method. This method can be efficiently used in the analysis of systems with small bandwidth, since the matrix decomposition must be executed at each step[2]. The subspace iteration method combines the simultaneous inverse iteration method and the Rayleigh-Ritz method. This method has been used

mostly, but the following shortcomings have been identified after extensive use of the method[4,5].

1) When the inverse iteration with shift is applied to the method to increase convergence, the shift value may be close to exact eigenvalue and numerical instability problems may be encountered during triangularization.

2) One considering the solution of eigenvalue problem for a relatively large number of eigenpairs, the solution time used in the subspace iteration method increases rapidly as the number of eigenpairs considered is increased. It is due to a number of factors that can be neglected when the solution of only a few eigenpairs is required.

① When p (p : the number of eigenpairs to be required) is large, the convergence rate of the eigenvector can be close to unity.

② If q is increased, the convergence of the smallest eigenvalues is generally achieved in a few iterations, and the converged vectors plus $(p+1)$th to qth iteration vectors are only included in the additional iterations to provide solution stability and to accelerate the convergence of the large required eigenvalues.

This paper, therefore, proposes an efficient solution method in order to improve numerical stability and increase convergence. As examples for calculating eigenvalues and the corresponding mode shapes, the plane frame and the three-dimensional building frame structure are analyzed to prove the efficiency of the proposed method.

2. METHOD OF ANALYSIS

Consider a generalized eigenvalue problem such as,

$$A\bar{x}_j = \lambda_j B\bar{x}_j \qquad (j = 1, 2, 3, \ldots, n) \tag{1}$$

where A and B are, respectively, the stiffness matrix and mass matrix of order n. B is positive definite and A positive semidefinite. λ_j is the jth natural frequency squared and $\bar{x}_j$ the corresponding mode shape.

Let us assume that initial approximate solutions of Eq. (1), $\lambda_j^{(0)}$ and $\bar{x}_j^{(0)}$, are available. Denote an approximate eigenvalue and the corresponding eigenvector after k iterations by $\lambda_j^{(k)}$ and $\bar{x}_j^{(k)}$ ($k = 0, 1, 2, \cdots$). Then, we have

$$\bar{r}_j^{(k)} = \bar{r}_j^{(k)}\,(\,\lambda_j^{(k)}\,,\,\bar{x}_j^{(k)}\,) = A\,\bar{x}_j^{(k)} - \lambda_j^{(k)}\,B\,\bar{x}_j^{(k)} \tag{2}$$

where the residual vector, $\bar{r}_j^{(k)}$, is not generally zero because of substitution of approximate values into Eq. (1). In order to have an approximate eigenvalue and the corresponding eigenvector converged to the eigenvalue and the corresponding eigenvector of the system, the residual vector should be removed. Let us apply the Newton-Raphson technique for this purpose.

$$\bar{r}_j^{(k+1)} = A\,\bar{x}_j^{(k+1)} - \lambda_j^{(k+1)}\,B\,\bar{x}_j^{(k+1)} \tag{3}$$

where

$$\lambda_j^{(k+1)} = \lambda_j^{(k)} + \Delta\lambda_j^{(k)} \tag{4}$$

$$\bar{x}_j^{(k+1)} = \bar{x}_j^{(k)} + \Delta\bar{x}_j^{(k)} \tag{5}$$

Substituting Eqs. (2), (4) and (5) into Eq. (3) and neglecting the higher order term $\Delta\lambda_j^{(k)}\,B\,\Delta\bar{x}_j^{(k)}$, we get

$$(\,A - \lambda_j^{(k)}\,B\,)\,\Delta\bar{x}_j^{(k)} - \Delta\lambda_j^{(k)}\,B\,\bar{x}_j^{(k)} = -\bar{r}_j^{(k)} \tag{6}$$

where $\Delta\lambda_j^{(k)}$ and $\Delta\bar{x}_j^{(k)}$ are incremental values of $\lambda_j^{(k)}$ and $\bar{x}_j^{(k)}$.

Because there are only n equations with $n+1$ unknowns which are $\Delta\lambda_j^{(k)}$ and n components of $\Delta\bar{x}_j^{(k)}$, a side condition must be introduced for the solution of Eq. (6). The side condition to formulate a set of $n+1$ equations with $n+1$ unknowns is

$$(\,\bar{x}_j^{(k)}\,)^T\,B\,\Delta\bar{x}_j^{(k)} = 0 \tag{7}$$

This is equivalent to saying that the allowable changes in the eigenvector are orthogonal to the latest eigenvector with respect to the mass matrix. This prevents unlimited drift in the eigenvector which is, after all, not determined in magnitude.

Writing Eq. (6) and Eq. (7) in a set of simultaneous linear algebraic equation, we get

$$\begin{bmatrix} A - \lambda_j^{(k)}\,B & -B\,\bar{x}_j^{(k)} \\ -(\,\bar{x}_j^{(k)}\,)^T\,B & 0 \end{bmatrix} \begin{Bmatrix} \Delta\bar{x}_j^{(k)} \\ \Delta\lambda_j^{(k)} \end{Bmatrix} = - \begin{Bmatrix} \bar{r}_j^{(k)} \\ 0 \end{Bmatrix} \tag{8}$$

The coefficient matrix for the incremental values is of order $n+1$ and symmetric. If λ_j's are not multiple, it is nonsingular[6].

The above algorithm using the Newton-Raphson method, despite of its rapid convergence, is expensive and inconvenient, since a new coefficient matrix has to be formed and refactorized in each iteration step. The complete elimination procedure in each iteration may be avoided by using the modified Newton-Raphson method in Eq. (8).

$$\begin{bmatrix} A-\lambda_j^{(0)}B & -B\,\bar{x}_j^{(k)} \\ -(\bar{x}_j^{(k)})^T B & 0 \end{bmatrix} \begin{Bmatrix} \Delta\bar{x}_j^{(k)} \\ \Delta\lambda_j^{(k)} \end{Bmatrix} = -\begin{Bmatrix} \bar{r}_j^{(k)} \\ 0 \end{Bmatrix} \tag{9}$$

The coefficient matrix in Eq. (9) is also nonsingular and symmetric.

The convergence rate in $\lambda_j^{(k)}$ and $\bar{x}_j^{(k)}$ of Eq. (9) using the modified Newton-Raphson method can be written as[6]

$$\kappa_j^{(k+1)} = h^2\,\kappa_j^{(k)} \tag{10}$$

$$\theta_j^{(k+1)} = h\,\theta_j^{(k)} \tag{11}$$

where $\kappa_j^{(k)} = \left|\dfrac{\lambda_j - \lambda_j^{(k)}}{\lambda_j}\right|$, and $h = \max\limits_{\substack{i \\ i\neq j}} \left|\dfrac{\lambda_j - \lambda_j^{(0)}}{\lambda_i - \lambda_j^{(0)}}\right|$. $\theta_j^{(k)}$ which represents the angle between $\bar{x}_j^{(k)}$ and $\bar{x}_j$ is an error in $\bar{x}_j^{(k)}$.

Once the submatrix $A-\lambda_j^{(0)}B$ is decomposed into the LDL^T (L : lower triangular matrix, D : diagonal matrix), a small number of operations are required for the solution of Eq. (9) in the succeeding iterations, since the vector $B\,\bar{x}_j^{(k)}$ in the coefficient matrix is only changed in each iteration. However, due to negligence of the small nonlinear term $(\lambda_j^{(k+1)}-\lambda_j^{(0)})\,B\,\Delta\bar{x}_j^{(k)}$, the convergence is lower. Thus, the number of iterations for a solution is increased. Some of these drawbacks may be avoided by the accelerated scheme such as,

$$\lambda_j^{(k+1)} = \lambda_j^{(k)} + \Delta\lambda_j^{(k)} \tag{12}$$

$$\bar{x}_j^{(k+1)} = \bar{x}_j^{(k)} + \alpha_j^{(k)}\,\Delta\bar{x}_j^{(k)} \tag{13}$$

$\alpha_j^{(k)}$ is a value to minimize the norm of the residual vector. It can be evaluated by using the least square method as follows;

$$\frac{\partial}{\partial \alpha_j^{(k)}} \{ (\bar{r}_j^{(k+1)})^T \bar{r}_j^{(k+1)} \} = 0 \tag{14}$$

$$\alpha_j^{(k)} = \frac{(\Delta \bar{x}_j^{(k)})^T (A - \lambda_j^{(k+1)} B)(A - \lambda_j^{(k+1)} B) \bar{x}_j^{(k)}}{(\Delta \bar{x}_j^{(k)})^T (A - \lambda_j^{(k+1)} B)(A - \lambda_j^{(k+1)} B) \Delta \bar{x}_j^{(k)}} \tag{15}$$

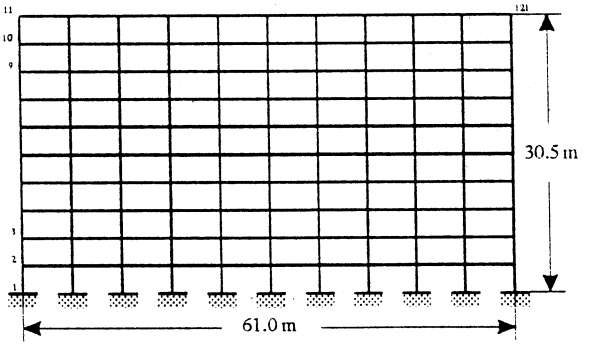
If the order of the system is n, and the bandwidths of the stiffness matrix and mass matrix are m_a and m_b, the number of operations for evaluating $\alpha_j^{(k)}$ in the first iteration step is $2nm_a+2nm_b+7n+1$. This is large compared to $5nm_a+2nm_b+6n$ which is required in each iteration step in Eq. (9). However, only the number of $7n+1$ operations is required to evaluate $\alpha_j^{(k)}$ after the 2nd iteration because we use additional outputs in the previous step. Thus, though it requires a few additional operations, solution time of the proposed method is decreased by improving convergence.

3. NUMERICAL EXAMPLES

The plane frame and the three-dimensional building frame which K. J. Bathe used[3] are analyzed to verify the efficiency of the proposed method. When the predetermined error norm is 1.E-09, the structures are analyzed by three different methods, the subspace iteration method, the determinant search method and the proposed method, and each convergence and solution time used to calculate 15 eigenpairs are compared. Intermediate results with relative error of 1.E-01 in the subspace iteration method are used as initial values of the proposed method. $\alpha_j^{(k)}$ is applied to the eigenpair whose error norm is over 1.E-01. All runs are executed in the IRIS4D-20-S17.

3.1 PLANE FRAME STRUCTURE

The plane frame structure which has 10-story and 10-bay shown in Fig. 1 consists of 210 beam elements, 121 nodes and 330 degrees-of-freedom. The maximum half-bandwidth of the stiffness matrix and mass matrix is 33 identically.



$A = 0.2787\ m^2$ $I = 8.631 \times 10^{-1}\ m^4$
$E = 2.068 \times 10^{10}\ Pa$ $\rho = 5.154 \times 10^2\ kg/m^3$

Fig. 1. Plane Frame Structure

Table 1. Solution Times(CPU, sec)

Methods	Plane Frame(Ratio)
Proposed Method	58.0 (1.0)
Subspace Iteration Method	155.4 (2.7)
Determinant Search Method	133.5 (2.3)

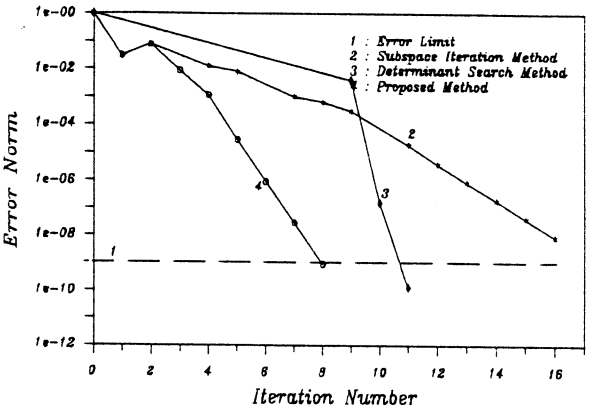


Fig. 3. Convergence of the 14th Eigenpair

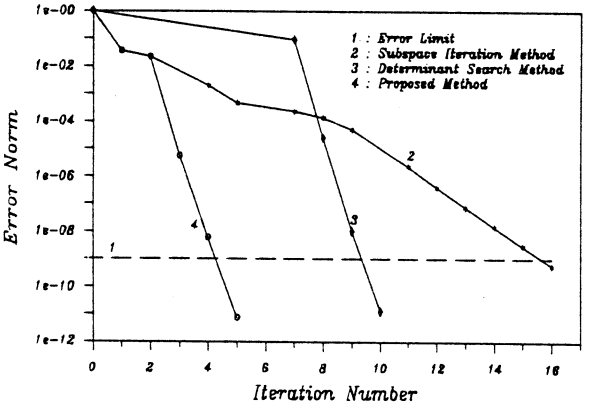


Fig. 2. Convergence of the 13th Eigenpair

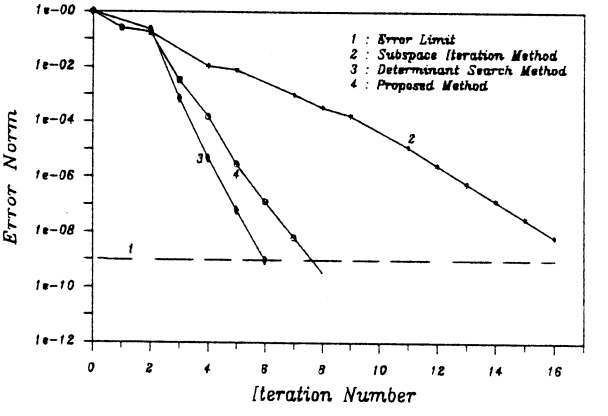


Fig. 4. Convergence of the 15th Eigenpair

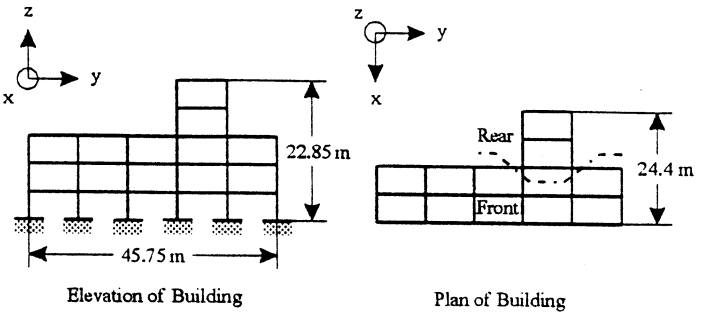
$\alpha_j^{(k)}$ is applied to the 13th, the 14th and the 15th eigenpair with error norm exceeding 1.E-01.

Solution times for three methods are summarized in Table 1. If we let the solution time for the proposed method be 1.0, it takes 2.68 times for the subspace iteration method, 2.3 times for the determinant search method. For each solution method, the convergence of eigenpairs to which $\alpha_j^{(k)}$ is applied is depicted in Fig. 2 ~ 4. According to them, it is obvious that the convergence of the proposed method is superior to that of the subspace iteration and of the determinant search method. $\alpha_j^{(k)}$ calculated in the above numerical example ranges 0.85 to 1.5.

3.2 THREE DIMENSIONAL BUILDING FRAME

Three dimensional building frame shown in Fig. 2 consists of 191 beam elements, 100 nodes and 468 degrees-of-freedom. The maximum half-bandwidth of the stiffness matrix and mass matrix is 138 identically. $\alpha_j^{(k)}$ is applied to the 12th, the 14th and the 15th eigenpair with error norm exceeding 1.E-01.

Solution times for three methods are summarized in Table 2.



Column in Front Building: A = 0.2787 m^2, I = 8.631×10^{-1} m^4
Column in Rear Building: A = 0.3716 m^2, I = 10.789×10^{-1} m^4
All Beams into x-Direction: A = 0.6096 m^2, I = 6.473×10^{-1} m^4
All Beams into y-Direction: A = 0.2787 m^2, I = 8.631×10^{-1} m^4
E = 2.068×10^{10} Pa, ρ = 5.154×10^{2} kg/m^3

Fig. 5. Three Dimensional Building Frame

Table 2. Solution Times(CPU, sec)

Methods	3-D. Frame(Ratio)
Proposed Method	217.4 (1.0)
Subspace Iteration Method	723.9 (3.3)
Determinant Search Method	1111.3 (5.1)

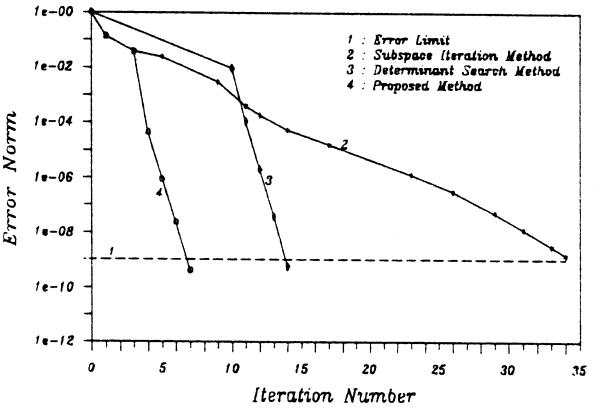


Fig. 7. Convergence of the 14th Eigenpair

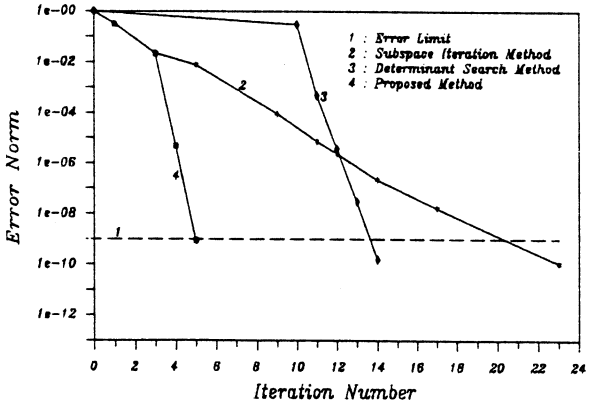


Fig. 6. Convergence of the 12th Eigenpair

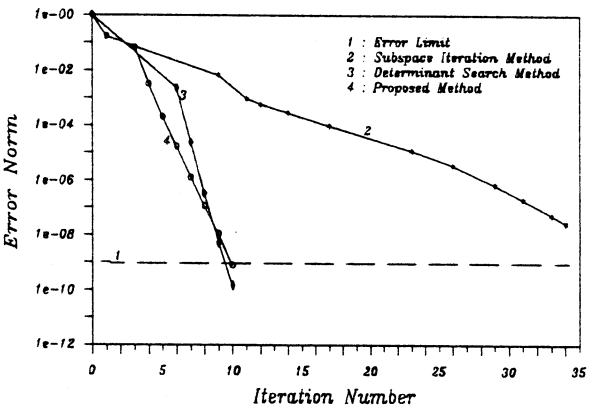


Fig. 8. Convergence of the 15th Eigenpair

If we let the solution time for the proposed method be 1.0, it takes 3.33 times for the subspace iteration method, 5.11 times for the determinant search method. For each solution method, the convergence of eigenpairs to which $\alpha_j^{(k)}$ is applied is presented in Fig. 6 ~ 8. $\alpha_j^{(k)}$ calculated in the above numerical example has the value of 0.85 to 1.02.

4. CONCLUSIONS

This paper proposes an efficient solution method using the accelerated Newton-Raphson method for eigenproblems. As shown in numerical examples of section 3, characteristics of the proposed method are identified as follows;

① Since each eigenpair is obtained independently, an eigenpair is not affected by eigenpairs previously calculated.

② The proposed method is numerically stable and converges very fast.

③ Missed eigenpairs can be detected with negligible operations and can be found by determinant search method.

Finally, this paper points out that the proposed method can be improved by applying an algorithm which can solve the eigenproblem having multiple roots.

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