Shift를 갖는 Subspace Iteration 방법의 개선

Improvement of Subspace Iteration Method with a Shift

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1. Introduction

Eigenvalue analysis is an important step in structural dynamic analysis when the mode superposition method is used. Many solution methods have been developed for eigenvalue analysis, and among these the subspace iteration method has hitherto been known to be very efficient for solving large eigenvalue problems.

The subspace iteration method was developed independently by Dong *et al.*[1] and Bathe and Wilson[2]. It is a direct-iterative method for symmetric matrices which combines inverse iteration, simultaneous iteration and Rayleigh-Ritz analysis. The basic objective in the subspace iteration method is to solve for the lowest p eigenvalues and corresponding eigenvectors satisfying

$$K X = M X \Lambda \tag{1}$$

where K and M are the stiffness matrix and the mass matrix of the discrete or discretized system of order n, respectively, the columns of X the eigenvectors, and Λ a diagonal matrix with eigenvalues.

This method has been widely used, but the following shortcomings[3] have been identified after extensive use of the method. These include: (1) slow convergence of the subspace for systems containing clusters of near-identical eigenvalues; (2) slow convergence and large computational and storage costs when a relatively large number of eigenpairs are required; and (3) missed eigenvectors caused by a poor choice of starting trial vectors. Due to these and other concerns, a variety of acceleration procedures have been developed.

Among the accelerated techniques, a shifting technique is effectively used in the commercial FEM programs such as ADINA.[4] Since the singularity may occur during the use of the shifting technique in the accelerated scheme such as the accelerated subspace iteration method, the shift must be within a limited region to avoid the singularity.

This paper describes a technique which always guarantees the numerical stability and maintains the convergence rate of the accelerated subspace iteration method.

2. Subspace Iteration Method with a Shift

The general eigenvalue problem of the structural dynamics may be written as in equation (1) in the preceding chapter.

Applying a shift μ to the general eigenvalue problem gives

$$(K - \mu M) X = M X \Omega \tag{2}$$

where

$$\Omega = \Lambda - \mu I \tag{3}$$

and I is the unit matrix.

The essential idea of the subspace iteration method is to iterate simultaneously with p linearly independent trial vectors, which initially span the starting subspace $E^{(1)}$, until subspace $E^{(\infty)}$ is spanned to sufficient accuracy. The eigenvectors, X, then form an M-orthonormal basis of the p-dimensional least-dominant subspace, $E^{(\infty)}$, of the matrices K and M[2].

The following algorithm is used to perform the subspace iterations with the shift μ.

Step 1.

$$(K - \mu M) \bar{X}^{(k+1)} = M X^{(k)} \quad (k = 1,2,\cdots) \tag{4}$$

where $\bar{X}^{(k+1)}$ and $X^{(k)}$ are the $(n \times q)$ matrices.

Step 2.

$$\bar{K}^{(k+1)} = \bar{X}^{(k+1)^T} (K - \mu M) \bar{X}^{(k+1)} \tag{5}$$

$$\bar{M}^{(k+1)} = \bar{X}^{(k+1)^T} M \bar{X}^{(k+1)} \tag{6}$$

Step 3.

$$\bar{K}^{(k+1)} Q^{(k+1)} = \bar{M}^{(k+1)} Q^{(k+1)} \Omega^{(k+1)} \tag{7}$$

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where $Q^{(k+1)}$ and $\Omega^{(k+1)}$ are the $(q \times q)$ matrices.

Step 4.(improved eigenpairs)

$$X^{(k+1)} = \bar{X}^{(k+1)} Q^{(k+1)} \quad (8)$$

$$\Lambda^{(k+1)} = \Omega^{(k+1)} + \mu I \quad (9)$$

Then, provided that the trial vectors in $X^{(1)}$ are not orthogonal to one of the required eigenvectors and assuming an appropriate ordering of the trial vectors, $\Lambda^{(k+1)}$ converges to Λ and $X^{(k+1)}$ converges to X as k approaches infinity.

The convergence rate of the subspace iteration method with a shift is

$$(\lambda_j - \mu)/(\lambda_{q+1} - \mu). \quad (10)$$

Considering the shift in the subspace iteration method, it is important to develop a stable and reliable solution scheme. A major drawback in the use of the shift is that if a shift is an eigenvalue itself or very close to it, all iteration vectors immediately converge to the eigenvector corresponding to that eigenvalue. The vectors can then not be orthogonalized any more and the iteration procedure becomes unstable. If the shift is very close to an eigenvalue, the last pivot element in the LDL^T factorization of the coefficient matrix becomes small compared with its original value. To avoid this singularity, that is, to guarantee the stability of the subspace iteration method with the shift, the following condition was adopted in the subspace iteration method[5];

$$1.01\,\lambda_{s-1} \le \mu \le 0.99\,\lambda_s \quad (11)$$

where λ_{s-1} is a (s-1)th eigenvalue and λ_s sth eigenvalue.

It means that a shift must be within a limited region. This is the significant disadvantage of the accelerated subspace iteration method using shifting technique. The purpose of this paper is to remove the limitation in equation (11) for choosing the value of a shift μ.

3. Proposed Method

Consider the simultaneous inverse iteration step(*Step 1*) in the algorithm of the subspace iteration method with the shift μ;

$$(K - \mu M)\bar{X}^{(k+1)} = M X^{(k)}. \quad (12)$$

Equation (12) can be rewritten as follows.

$$(K - \mu M)\bar{X}^{(k+1)} = M X^{(k)} D^{(k)} \quad (13)$$

where $D^{(k)} = d_i^{(k)} (i = 1,2,\ldots,q)$ is the diagonal matrix of order q. The elements of $D^{(k)}$ control the lengths of the vectors in $\bar{X}^{(k+1)}$.

Because there are only $(n \times q)$ equations with $((n+1) \times q)$ unknowns, $(n \times q)$ components of $\bar{X}^{(k+1)}$ and q components of $D^{(k)}$, side conditions must be introduced for the solution of equation (13). These conditions are that the incremental vectors $\Delta X^{(k)}$ are orthogonal to the current vectors with respect to M; that is,

$$X^{(k)^T} M \Delta X^{(k)} = 0. \quad (14)$$

Adding the orthogonality condition, $X^{(k)^T} M X^{(k)} = I_q$, to the side conditions(see equation (14)) yields

$$X^{(k)^T} M (X^{(k)} + \Delta X^{(k)}) = X^{(k)^T} M \bar{X}^{(k+1)} = I_q \quad (15)$$

where

$$\bar{X}^{(k+1)} = X^{(k)} + \Delta X^{(k)} \quad (16)$$

Writing equations (13) and (15) in matrix form gives

$$\begin{bmatrix} K - \mu M & -MX^{(k)} \\ -X^{(k)^T} M & 0 \end{bmatrix} \begin{bmatrix} \bar{X}^{(k+1)} \\ D^{(k)} \end{bmatrix} = \begin{bmatrix} 0 \\ I_q \end{bmatrix} \quad (17)$$

where 0 is $(n \times q)$ null matrix and I_q unit matrix of order q. Note that $\bar{X}^{(k+1)}$ from equation (17) is used for $\bar{X}^{(k+1)}$ in equations (5) and (6) instead of $\bar{X}^{(k+1)}$ from equation (4). Equation (17) is the main linear algebraic equation used in the proposed method.

The coefficient matrix of equation (17) is of order $(n+q)$, symmetric and nonsingular. The nonsingularity[6] is one of the significant advantages of the proposed method.

We can verify that provided the starting subspace is not orthogonal to the required least dominant subspace spanned by $\phi_1, \phi_2, \ldots, \phi_q$, the jth column in $X^{(k+1)}$ converges linearly with the rate $(\lambda_j - \mu)/(\lambda_{q+1} - \mu)$ to ϕ_j.[7] Since the eigenvalues are calculated using the Rayleigh quotient, the jth eigenvalue in equation (7)

converges linearly with the rate $\left\{(\lambda_j - \mu)/(\lambda_{q+1} - \mu)\right\}^2$ to λ_j.

The number of operations for the accelerated subspace iteration method and the proposed method is almost same for large structures.

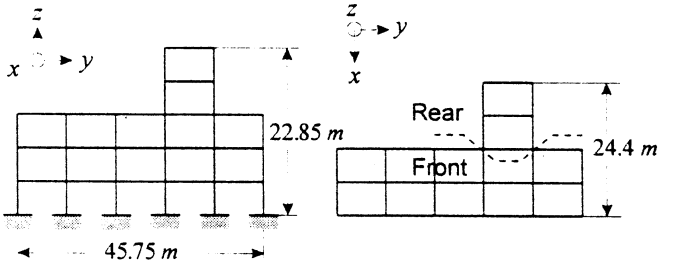
4. Numerical Example

The three-dimensional framed structure used by Bathe and Wilson[2] is analyzed to verify the effectiveness of the proposed method. With the predetermined relative error of 10^{-9}, the structures are analyzed by two methods; the accelerated subspace iteration method and the proposed method, where the relative error is computed by the following equation.

$$\text{relative error} = \left| \frac{\lambda_j^{(k+1)} - \lambda_j^{(k)}}{\lambda_j^{(k+1)}} \right| \qquad (18)$$

Even if a shift is on or very close to an eigenvalue, it is shown that the proposed method can obtain the solutions without any singularity. When a shift is not close to an eigenvalue, each convergence rate for calculating the first five or ten eigenpairs is compared. All runs are executed in the IRIS4D-20-S17 with 10 Mips and 0.9 Mflops.

4.1 Three-dimensional framed structure



Column in Front : $A = 0.2787\ m^2$, $I = 8.631\times10^{-3}\ m^4$

Column in Rear : $A = 0.3716\ m^2$, $I = 10.787\times10^{-3}\ m^4$

Beams(x - Dir.) : $A=0.6096m^2$, $I=6.473 \times 10^{-1} m^4$

Beams(y - Dir.) : $A = 0.2787\ m^2$, $I = 8.631\times10^{-1}\ m^4$

$E = 2.068\times10^{10}\ Pa$, $\rho = 5.154\times10^{2}\ kg/m^3$

Figure 1. Three-dimensional framed structure

Figure 1 shows the geometric configuration and material properties. The structure discretized using 100 beam elements resulting in system of dynamic equations with a total of 468 degrees of freedom. The consistent mass matrix is used for M.

Some results are shown in Table 1 and in Figures 2 to 4. The lowest ten eigenvalues are listed in Table 1. These eigenvalues are calculated by the proposed method when the shift is on the sixth eigenvalue of the three-dimensional framed structure, that is, $\mu = \lambda_6$. These are the same values that are evaluated by the standard subspace iteration method. Since the accelerated procedure without the limit condition(see equation (11)) can not find the solutions in this case, it is compared with the standard procedure having the low convergence rate relative to the accelerated procedure.

It shows that the iteration procedure of the proposed method can converge without the singularity even if the shift is the same exactly to an eigenvalue.[6]

For each solution method, the convergence of each eigenpair is depicted in Figures 2 to 4, when the shift is 2.0 ($\lambda_5 < \mu < \lambda_6$). From these results it is obvious that the convergence of the proposed method is nearly equal to that of the accelerated subspace iteration method.

5. Conclusions

This paper proposes a stable acceleration technique using side conditions for the improvement of the accelerated subspace iteration method with the shift. The characteristics of the proposed method identified by the analytical and the numerical results from a numerical example are summarized as follows:

① The nonsingularity of the proposed method is always guaranteed, which is proved analytically; even if the shift is an eigenvalue itself, the proposed method can obtain the solutions without any singularity.

② The convergence rate of the proposed method is at least equal to that of the accelerated subspace iteration method, and the operation counts of the proposed method and the

accelerated subspace iteration method are almost same for large structures.

References

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Table 1. *The lowest ten eigenvalues of the three-dimensional framed structure*

Eigenvalue number	Eigenvalue
1	0.4239
2	0.5504
3	0.8385
4	1.0703
5	1.6004
6	2.1468
7	2.5270
8	3.3019
9	3.7551
10	4.3716

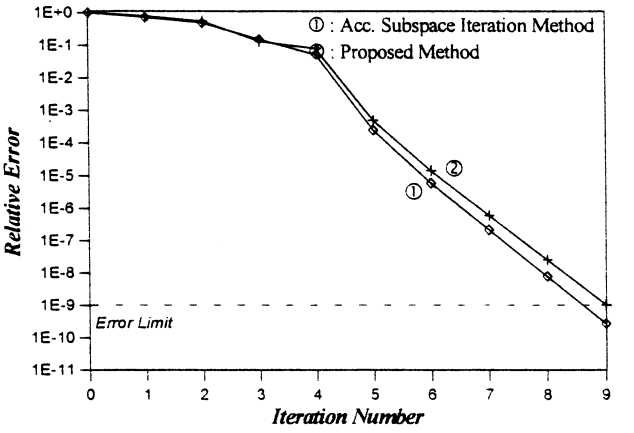

Figure 2. Relative error versus iteration number of the first eigenpair

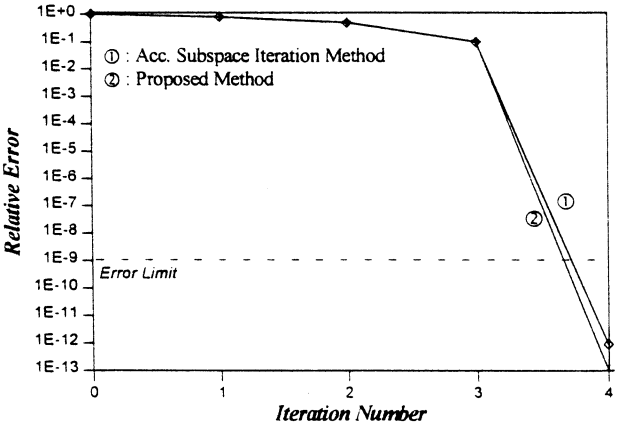

Figure 3. Relative error versus iteration number of the sixth eigenpair

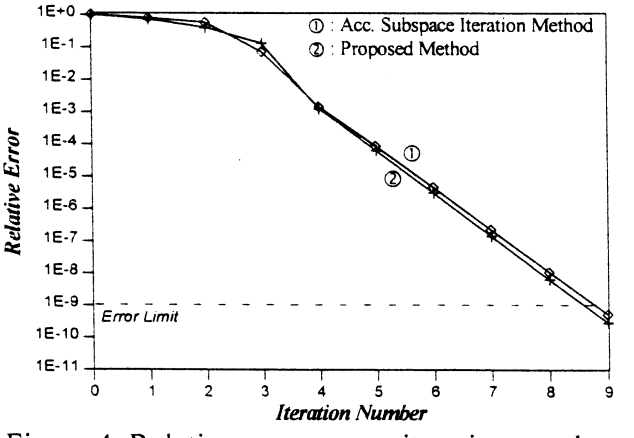

Figure 4. Relative error versus iteration number of the tenth eigenpair