An Efficient Solution Method of Generalized Eigenproblems with Multiple Eigenvalues

°Man-Cheol Kim[1], Hyung-Jo Jung[1], Ju-Won Oh[2] and In-Won Lee[3]

1. INTRODUCTION

In the analysis of dynamic response of structural systems, the equation of motion of damped systems can be written as

$$M\,\ddot{u}(t) + C\,\dot{u}(t) + K\,u(t) = f(t)\,, \tag{1}$$

where M, K and C are the (n by n) mass, stiffness and nonproportional damping matrices, respectively, and $\ddot{u}(t)$, $\dot{u}(t)$ and $u(t)$ are the (n by 1) acceleration, velocity and displacement vectors, respectively. To find the solution of free vibration of the system, we consider the following quadratic eigenproblem

$$\lambda^2 M\,\phi + \lambda\, C\,\phi + K\,\phi = 0 \tag{2}$$

in which λ and ϕ are the eigenvalue and eigenvector of the system. There are $2n$ eigenvalues for the system with n degrees of freedom and these occur either in real pairs or in complex conjugate pairs, depending upon whether they correspond to overdamped or underdamped modes.

The common practice is to reformulate the quadratic system of equation to a linear one by doubling the order of the system such as

$$A\,\psi = \lambda\, B\,\psi \tag{3}$$

where

$$A = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix},\quad B = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \text{ and } \psi = \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix} \tag{4}$$

2. METHOD OF ANALYSIS

2.1 Problem Definition

We consider an eigenproblem of which the eigenvalue λ_i has multiplicity m. For simplicity let us assume that the first m eigenvalues are equal

$$\lambda \equiv \lambda_1 = \lambda_2 = \cdots\cdots = \lambda_m\,. \tag{5}$$

Then Eq. (3) can be presented in matrix form for the m multiple eigenvalues as follows

$$A\,\Psi = B\,\Psi\,\Lambda \tag{6}$$

1) Graduate Student, Department of Civil Engineering, KAIST
2) Professor, Department of Civil and Environmental Engineering, Han-Nam University
3) Professor, Department of Civil Engineering, KAIST

where $\Lambda = diag(\lambda_1, \cdots, \lambda_m) = \lambda I_m$ and $\Psi = [\psi_1 \ \cdots \ \psi_m]$ is a (n by m) matrix satisfying the orthonormal condition with respect to matrix B such as

$$\Psi^T B \Psi = I_m \tag{7}$$

where I_m is an unitary matrix of order m.

2.2 Proposed Method

Let us assume that initial approximate solutions of Eq. (6) $\Lambda^{(0)}$ and $\Psi^{(0)}$ are known. Denoting the approximate eigenvalues and the associated eigenvectors after k iterations by $\Lambda^{(k)}$ and $\Psi^{(k)}$, we can get

$$R^{(k)} = A \Psi^{(k)} - B \Psi^{(k)} \Lambda^{(k)} \tag{8}$$

and

$$(\Psi^{(k)})^T B \Psi^{(k)} = I_m \tag{9}$$

where the residual matrix $R^{(k)} = [r_1^{(k)} \ \cdots \ r_m^{(k)}]$ denotes the error for each eigenpair, and is not generally zero because of substitution of approximate values into Eq. (6).

In order to get the solutions converged to the multiple eigenvalues and the associated eigenvectors of the system, the residual vectors should be removed. For the purpose of that, the Newton-Raphson technique is applied such as

$$\begin{aligned} R^{(k+1)} &= A \Psi^{(k+1)} - B \Psi^{(k+1)} \Lambda^{(k+1)} \\ &= 0 \end{aligned} \tag{10}$$

and

$$(\Psi^{(k+1)})^T B \Psi^{(k+1)} = I_m \tag{11}$$

where

$$\Lambda^{(k+1)} = \Lambda^{(k)} + \Delta\Lambda^{(k)} \tag{12}$$

$$\Psi^{(k+1)} = \Psi^{(k)} + \Delta\Psi^{(k)}. \tag{13}$$

Substituting Eq. (12) and Eq. (13) into Eq. (10) and Eq. (11) and neglecting the nonlinear terms $B \Delta\Psi^{(k)} \Delta\Lambda^{(k)}$ and $(\Delta\Psi^{(k)})^T B \Delta\Psi^{(k)}$, we can get the linear simultaneous equations for unknown incremental values $\Delta\Lambda^{(k)}$ and $\Delta\Psi^{(k)}$ as follows;

$$A \Delta\Psi^{(k)} - B \Delta\Psi^{(k)} \Lambda^{(k)} - B \Psi^{(k)} \Delta\Lambda^{(k)} = - R^{(k)} \tag{14}$$

and

$$(\Psi^{(k)})^T B \Delta\Psi^{(k)} = 0. \tag{15}$$

Since the eigenvalue is multiple, the off-diagonal elements of $\Lambda^{(k)}$ are zero or very small compared with its diagonal at kth iteration step, and the diagonal element very close. Thus, the second term in left side of Eq. (14) may be approximated by $\lambda_1^{(k)} B \Delta\Psi^{(k)}$, which yields

$$A \Delta\Psi^{(k)} - \lambda_1^{(k)} B \Delta\Psi^{(k)} - B \Psi^{(k)} \Delta\Lambda^{(k)} = - R^{(k)} \tag{16}$$

Writing Eq. (16) and Eq. (15) in matrix form, we can get

$$\begin{bmatrix} (A-\lambda_1^{(k)}B) & -B\Psi^{(k)} \\ (-B\Psi^{(k)})^T & 0 \end{bmatrix} \begin{bmatrix} \Delta\Psi^{(k)} \\ \Delta\Lambda^{(k)} \end{bmatrix} = -\begin{bmatrix} R^{(k)} \\ 0 \end{bmatrix} \tag{17}$$

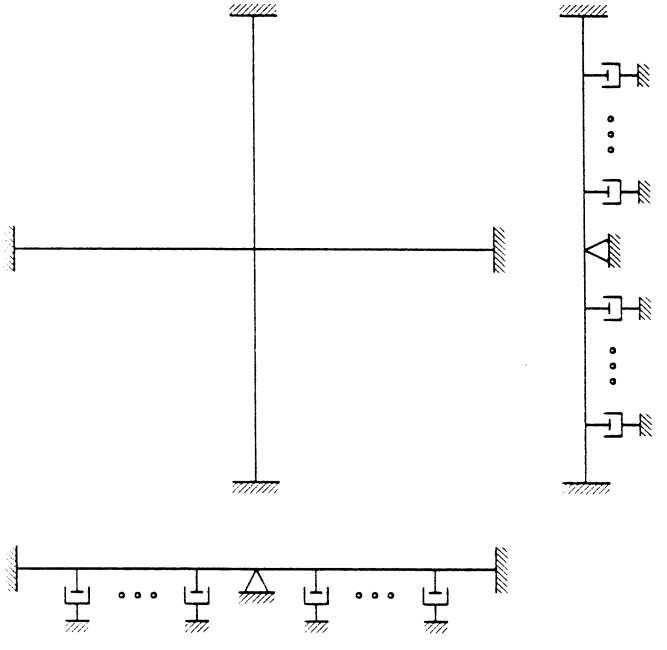
Because the new coefficient matrix should be reformed and refactorized in each iteration step, the above method adopting the Newton-Raphson technique and a side condition, despite of its rapid convergence, is not efficient.

These blemishes may be overcome by applying the modified Newton-Raphson technique to Eq. (17) such as

$$\begin{bmatrix} (A-\lambda_1^{(0)}B) & -B\Psi^{(k)} \\ (-B\Psi^{(k)})^T & 0 \end{bmatrix} \begin{bmatrix} \Delta\Psi^{(k)} \\ \Delta\Lambda^{(k)} \end{bmatrix} = -\begin{bmatrix} R^{(k)} \\ 0 \end{bmatrix} \tag{18}$$

The symmetric coefficient matrix of Eq. (18) is of order $(2n+m)$. While singularity occurs in factorization process of the iteration methods such as the inverse iteration method and the subspace iteration method when the shift is close to an eigenvalue of the system, nonsingularity of the proposed method is always guaranteed. This is the main difference compared with the iteration method with shift.

3. NUMERICAL EXAMPLE



Young's Modulus :1000 *Mass Density :1.0*
Cross-section Inertia:1.0 *Cross-section Area:1.0*
ProportionalDamping Coeff.: $\alpha = 0.001$, $\beta = 0.001$
Concentrated Damping :0.3

Fig.1 Plane frame structure with lumped dampers

In this section a test problem with multiple eigenvalues is used to assess the performance of the proposed method for generalized eigenproblems. The CPU time spent for the first twelve eigenvalues and the associated eigenvectors (p=12) and the variation of the error norm to each iteration step of the proposed method are compared with those of the subspace iteration method. The least subspace dimension to effectively calculate required eigenpairs is $2p$ (24 eigenpairs). Each method is stopped when the error norms are reduced by the factor of 10^{-6}, which yields a stable eigensolution and sufficient accuracy in the calculated eigenvalues and eigenvectors for practical analysis. The error norm is defined as

$$\varepsilon_i^{(k)} = \frac{\left\| r_i^{(k)} \right\|_2}{\left\| A\,\psi_i^{(k)} \right\|_2} \tag{19}$$

where

$$R^{(k)}=\left[r_1^{(k)} \quad \cdots \quad r_m^{(k)}\right]=A\Psi^{(k)}-B\Psi^{(k)}\Lambda^{(k)} \tag{20}$$

3.1 Plane Frame Structure with Lumped Dampers

The finite element model of a plane frame is used as the first example. The geometric configuration and material properties are shown in Fig. 1. The model is discretized in 200 beam elements resulting in the system of dynamic equations with a total of 590 degrees of freedom. Thus, the order of the associated eigenproblem is 1180. The consistent mass matrix is used for M. Its damping matrix is derived from the proportional damping expression given by $C = \alpha K + \beta M$ and concentrated dampers.

The initial values of the proposed method obtained by using the $4p$ (48) Lanczos vectors have the error norms of about 0.7 to 10^{-7}, which are possible to check the multiplicity of the desired eigenvalues. The proposed method using the above values as initial values needs only one iteration to satisfy the error norm 10^{-6} but the subspace iteration method 49 iterations. The results indicate that the convergence of the proposed method is much better than that of the subspace iteration method. The results of the Lanczos method are not improved in spite of the increase of the number of the Lanczos vectors. The CPU time for the proposed method is compared with that of the subspace iteration method in Table 1. If we let the solution time for the proposed method be 1, it takes 3.55 times for the subspace iteration method.

Table 1. CPU time spent for the first twelve eigenvalues of the plane frame structure with lumped dampers

Methods	CPU time(in seconds), Ratio
Proposed method	872.69 (1.00)
Subspace iteration method	3096.62 (3.55)

4. CONCLUSIONS

An efficient method for solving damped structural dynamic eigenproblems with multiple eigenvalues is presented. Characteristics of the proposed method identified by the numerical results from a test problem are identified as follows;

① Since the convergence rate of the proposed method is high, the proposed method is very effective for solving damped dynamic systems with a large number of degrees of freedom.

② Nonsingularity of the proposed method is always guaranteed, which is proved analytically.

③ The algorithm of the proposed method is very simple.

5. REFERENCES

Leung, A.Y.T. (1995), "Subspace iteration method for complex symmetric eigen-problems", *J. Sound Vibr.*, 184(4), 627-637.

Chen, H.C. and Taylor, R.L.(1988), "Solution of eigenproblems for damped structural systems by Lanczos algorithm", *Comput. Struct.*, 30(1/2), 151-161.