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Free Vibration Analysis of Non-Proportionally Damped Structures with Multiple or Close Frequencies

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ABSTRACT

An efficient solution method is presented to solve the eigenvalue problem arising in the dynamic analysis of non-proportionally damped structural systems with multiple or close eigenvalues. The proposed method is obtained by applying the modified Newton-Raphson technique and the orthonormal condition of the eigenvectors to the quadratic eigenvalue problem. Even if the shift value is an eigenvalue of the system, the proposed method guarantees nonsingularity, which is analytically proved. The initial values of the proposed method can be taken as the intermediate results of iteration methods or results of approximate methods. Two numerical examples are also presented to demonstrate the effectiveness of the proposed method and the results are compared with those of the well-known subspace iteration method and the Lanczos method.

1. Introduction

To find the solution of free vibration of the non-proportional damping system, we consider the following quadratic eigenproblem

$$\lambda^2 M \phi + \lambda C \phi + K \phi = 0 \tag{1}$$

where M, K and C are the (n by n) mass, stiffness and non-proportional damping matrices, respectively. λ and ϕ are the eigenvalue and eigenvector of the system. The common practice is to reformulate the quadratic system of equation to a linear one by doubling the order of the system such as

$$A \psi = \lambda B \psi \tag{2}$$

where

$$A = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \quad B = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \text{ and } \psi = \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix}. \tag{3}$$

The classical inverse iteration method[1] is commonly used to solve for only a small number of desired modes. The subspace iteration method[2] is a more efficient alternative than the inverse iteration method. However, in these methods, a large number of complex arithmetic operations are required. Furthermore, when the shift value becomes close to an eigenvalue of the system, singularity may be encountered during

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triangularization process.

The Lanczos algorithm for the computation of eigenvalues and eigenvectors of a real symmetric matrix was proposed by Lanczos in 1950 and improved by numerous researchers[3]. The Lanczos algorithm to solve the eigenvalue problem of non-proportionally damped system is developed in References [4] (two sided-Lanczos algorithm) and [5] (symmetric Lanczos algorithm). Although only real arithmetic is solved during the solution process, in contrast to the case of real symmetric eigenproblems, there will be a possibility of serious breakdown and the accuracy of the solutions obtained is low[6].

In this paper, the method to solve an eigenproblem with guaranteed nonsingularity for a non-proportionally damped structural system with close or multiple eigenvalues is developed[7].

2. Method of Analysis

2.1 Problem definition

We consider an eigenproblem of which the m eigenvalues are close or multiple. For simplicity let us assume that the first m eigenvalues are close or multiple.

$$\lambda_1 \cong \lambda_2 \cong \cdots\cdots \cong \lambda_m . \tag{4}$$

Then Eq. (2) can be presented in matrix form for the m eigenvalues as follows

$$A\,\Psi = B\,\Psi\,\Lambda \tag{5}$$

where $\Lambda = diag(\,\lambda_1, \cdots, \lambda_m\,)$, and $\Psi = [\,\psi_1 \quad \cdots \quad \psi_m\,]$ is a (n by m) matrix satisfying the orthonormal condition with respect to matrix B such as

$$\Psi^T\, B\,\Psi = I_m \tag{6}$$

where I_m is an unitary matrix of order m.

Let $X = [x_1, ..., x_m]$ be the vectors in the subspace Ψ, and X be the orthonormal with respect to matrix B. Then

$$\Psi = XZ \tag{7}$$

$$X^T BX = I_m \tag{8}$$

where Z is the unknown rotation matrix of order m.

Introducing Eq. (7) into Eq. (5), we get

$$AXZ = BXZ\Lambda . \tag{9}$$

Let

$$DZ = Z\Lambda \tag{10}$$

where $D = [d_1, d_2, ..., d_m] = X^T AX$ and symmetric.

Then,

$$AXZ = BXDZ \tag{11}$$

and

$$AX = BXD \tag{12}$$

or

$$Ax_i = BXd_i \qquad (i = 1,...,m). \tag{13}$$

We obtain the m close or multiple eigenvalues and associated eigenvectors from Eqs. (7), (10) and (13). Note when $\lambda_1 = ... = \lambda_m$, from Eq. (12)

$$D = \Lambda \tag{14}$$

$$X = \Psi \tag{15}$$

The objective is to develop an efficient solution method with guaranteed non-singularity for an eigenproblem described by Eq. (13).

2.2 Proposed Method

Let us assume that initial approximate solutions of Eq. (13), $d_i^{(0)}$ and $x_i^{(0)}$, are known. Denoting the approximate eigenvalues and the associated eigenvectors after k iterations by $d_i^{(k)}$ and $x_i^{(k)}$, we can get

$$r_i^{(k)} = Ax_i^{(k)} - BX^{(k)} d_i^{(k)} \tag{16}$$

and

$$(X^{(k)})^T BX^{(k)} = I_m \tag{17}$$

where the residual vector, $r_i^{(k)}$, denotes the error for each eigenpair, and is not generally zero because of substitution of approximate values into Eq. (13).

In order to get the solutions converged to the multiple or close eigenvalues and the associated eigenvectors of the system, the residual vectors should be removed. For the purpose of that, the Newton-Raphson technique is applied such as

$$\begin{aligned} r_i^{(k+1)} &= Ax_i^{(k+1)} - BX^{(k+1)} d_i^{(k+1)} \\ &= 0 \end{aligned} \tag{18}$$

and

$$(X^{(k+1)})^T BX^{(k+1)} = I_m \tag{19}$$

where

$$d_i^{(k+1)} = d_i^{(k)} + \Delta d_i^{(k)} \tag{20}$$

$$x_i^{(k+1)} = x_i^{(k)} + \Delta x_i^{(k)} \tag{21}$$

where $X^{(k+1)} = [x_1^{(k+1)}, x_2^{(k+1)}, ..., x_m^{(k+1)}]$, and $\Delta d_i^{(k)}$ and $\Delta x_i^{(k)}$ are unknown incremental values of $d_i^{(k)}$ and $x_i^{(k)}$.

Substituting Eqs. (16), (17), (20) and (21) into Eqs. (18) and (19) and neglecting the nonlinear terms, we can get the linear simultaneous equations for unknown incremental values, $\Delta d_i^{(k)}$ and $\Delta x_i^{(k)}$, as follows:

$$A\Delta x_i^{(k)} - B\Delta X^{(k)} d_i^{(k)} - BX^{(k)} \Delta d_i^{(k)} = -r_i^{(k)} \tag{22}$$

and

$$(X^{(k)})^T B\Delta X^{(k)}=0. \tag{23}$$

Since the eigenvalue is multiple or close, the off-diagonal elements of D are zero or very small compared with its diagonal at k th iteration step, and the diagonal elements very close. Thus, the second term in left side of Eq. (22) may be approximated by $d_{ii}^{(k)} B\Delta x_i^{(k)}$, which yields

$$A\Delta x_i^{(k)}-d_{ii}^{(k)} B\Delta x_i^{(k)}-BX^{(k)}\Delta d_i^{(k)}=-r_i^{(k)}. \tag{24}$$

Writing Eqs. (23) and (24) in matrix form, we can get

$$\begin{bmatrix} (A-d_{ii}^{(k)} B) & -BX^{(k)} \\ (-BX^{(k)})^T & 0 \end{bmatrix} \begin{Bmatrix} \Delta x_i^{(k)} \\ \Delta d_i^{(k)} \end{Bmatrix} = -\begin{Bmatrix} r_i^{(k)} \\ 0 \end{Bmatrix} \qquad (i=1,\ldots,m) \tag{25}$$

Because the new coefficient matrix should be reformed and refactorized in each iteration step, the above method adopting the Newton-Raphson technique and a side condition, despite of its rapid convergence, is not efficient.

These blemishes may be overcome by applying the modified Newton-Raphson technique to Eq. (25) such as

$$\begin{bmatrix} (A-d_{ii}^{(0)} B) & -BX^{(k)} \\ (-BX^{(k)})^T & 0 \end{bmatrix} \begin{Bmatrix} \Delta x_i^{(k)} \\ \Delta d_i^{(k)} \end{Bmatrix} = -\begin{Bmatrix} r_i^{(k)} \\ 0 \end{Bmatrix} \qquad (i=1,\ldots,m) \tag{26}$$

The symmetric coefficient matrix of Eq. (26) is of order $(2n+m)$. While singularity occurs in factorization process of the iteration methods such as the inverse iteration method[1] and the subspace iteration method[2] when the shift is close to an eigenvalue of the system, nonsingularity of the proposed method is always guaranteed. This is the main difference compared with the iteration method with shift.

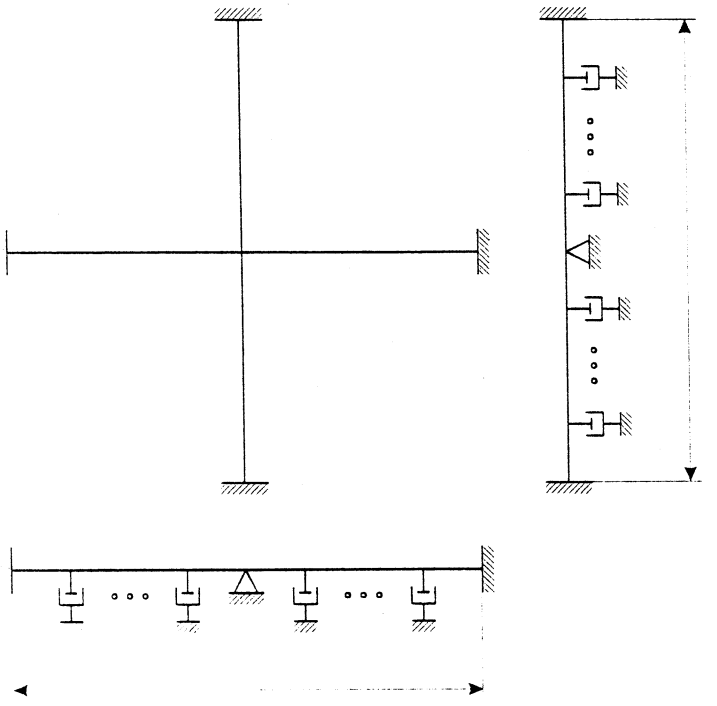
Initial values of the proposed method can be obtained as the intermediate results of the iteration methods or results of approximate methods. In this paper, the starting values are taken as the results of the symmetric Lanczos method[5] with selectively reorthogonalization process because the method does not need complex arithmetic in the Lanczos recursive process, and because the multiplicity of the desired eigenvalues can be checked by the results of the $4p$ Lanczos vectors(p : the number of desired eigenvalues).

3. Numerical Example

The CPU time spent for the first twelve eigenvalues and the associated eigenvectors (p=12) and the variation of the error norm to each iteration step of the proposed method are compared with those of the subspace iteration method. The least subspace dimension to effectively calculate required eigenpairs by the subspace iteration method is $2p$ (24 eigenpairs).

3.1 Plane Frame Structure with Lumped Dampers(Multiple Case)

The model is discretized in 200 beam elements resulting in the system of dynamic equations with a total of 590 degrees of freedom. Thus, the order of the associated eigenproblem is 1180. The consistent mass matrix is used for M. Its damping matrix is derived from the proportional damping expression given by $C=\alpha M+\beta K$ and concentrated dampers.



YOUNG'S MODULUS :	1,000
MASS DENSITY :	1.0
CROSS-SECTION INERTIA :	1.0
CROSS-SECTION AREA :	1.0
DAMPING COEFFICIENTS	
: $\alpha = 0.001$, $\beta = 0.001$	
CONCENTRATED DAMPING :	0.3

Fig. 1. Plane frame structure with lumped dampers

All the eigenvalues of the model are multiple. The variations of the error norms to increasing the iteration step are shown in Figs 2 to 4. The error norms of the initial values obtained by using the $4p$ (48) Lanczos vectors are about 0.7 to 10^{-7}, which are possible to check the multiplicity of the desired eigenvalues. The number of iterations for the proposed method applied to the initial values that do not satisfy the error norm 10^{-6} is only one. The results in Figs 2 to 3 indicate that the convergence of the proposed method is much better than that of the subspace iteration method. The results of the Lanczos method in Fig. 4 are not improved in spite of the increase of the number of the Lanczos vectors. The CPU of the proposed method is compared with that of the subspace iteration method in Table 1. If we let the solution time for the proposed method be 1, it takes 3.55 times for the subspace iteration method.

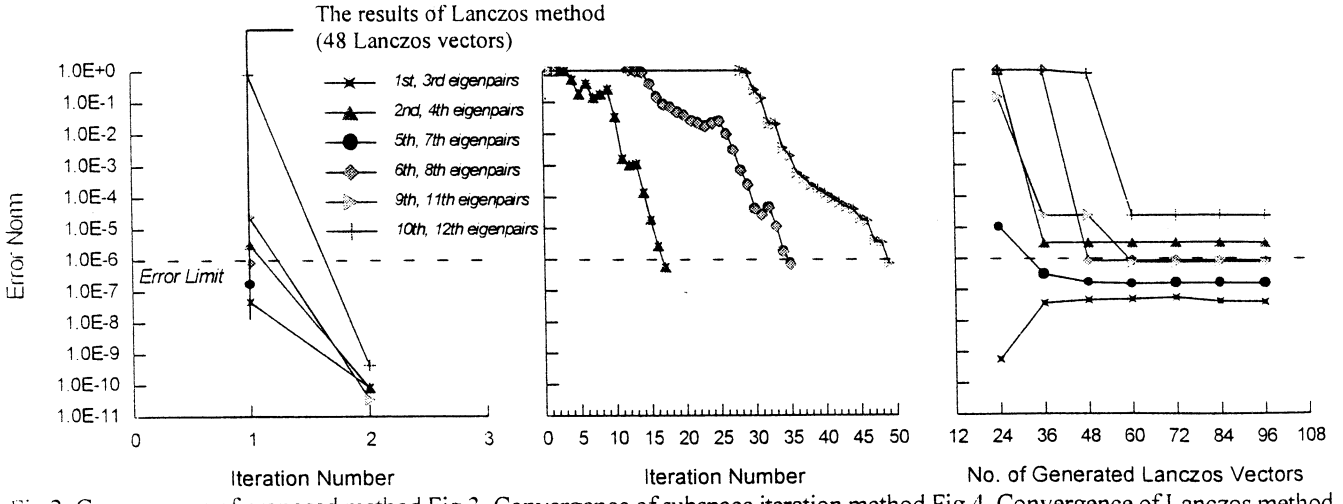


Fig.2. Convergence of proposed method Fig.3. Convergence of subspace iteration method Fig.4. Convergence of Lanczos method

Table 1. CPU time (in seconds) spent for the first twelve eigenvalues of the model

Methods	CPU time (Ratio)
Proposed method	872.69 (1.00)
Subspace iteration method	3096.62 (3.55)

3. 2 Three-Dimensional Building Structure with Concentrated Dampers(Close Case)

In this example a three-dimensional building structure with concentrated dampers is presented. The geometric configuration and material properties are shown in Fig. 5. The model is divided into 436 beam elements and has 1,128 degrees of freedom. The order of the associated eigenproblem is 2,256. The consistent mass matrix is used to define M. The damping matrix consists of the Rayleigh damping and concentrated dampers.

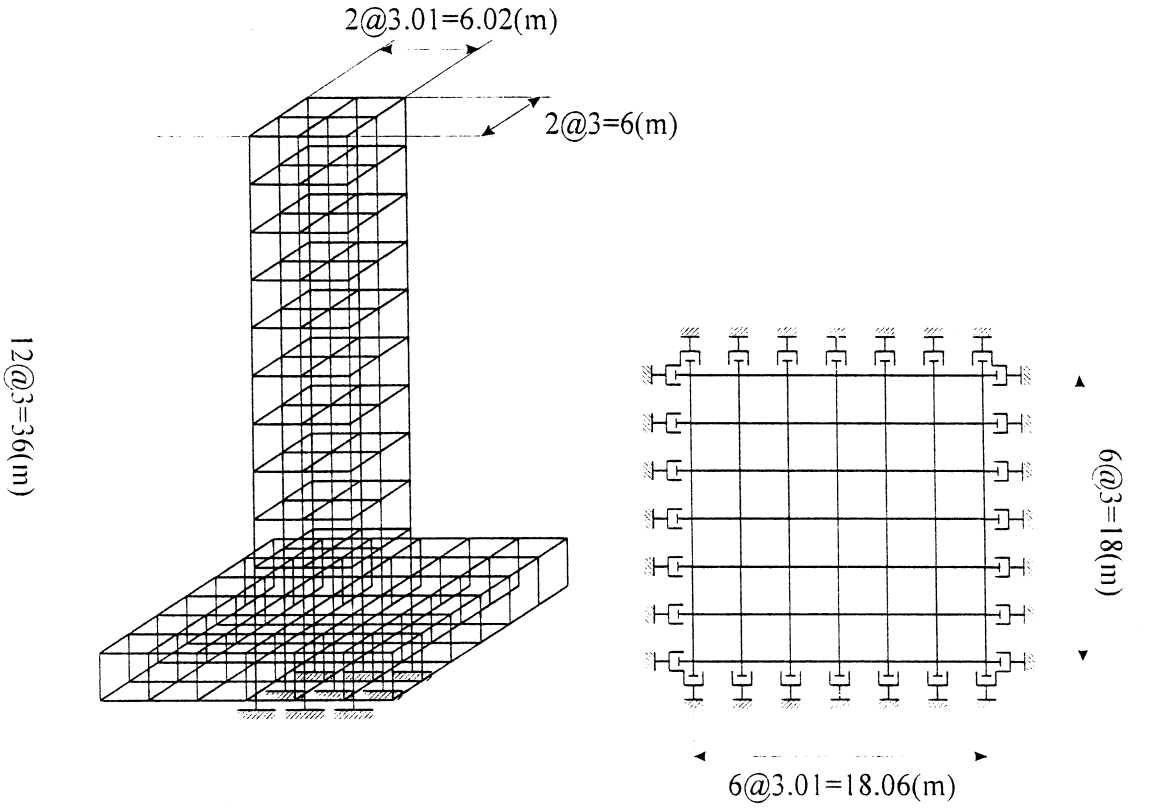
The results of the proposed method are summarized in Table 2. The first and third eigenvalue are clustered, and also the ninth and eleventh eigenvalues and their conjugate eigenvalues clustered. The variations of the error norms to the iteration step are shown in Figs 6 to 8. The first step of the proposed method denotes the results of the Lanczos algorithm. The error norms of the initial values obtained by using the 48 Lanczos vectors are about 0.9 to 10^{-6}. The number of iterations for the proposed method applied to initial values that do not satisfy the error norm 10^{-6} is three or eleven. The results in Figs 6 to 8 indicate that the convergence of the proposed method is much better than that of the subspace iteration method. The results of the Lanczos method in Fig. 8 are not improved in spite of the increase of the number of the Lanczos vectors. The CPU time for the proposed method is compared with the subspace iteration method in Table 3. If we let the solution time for the proposed method be 1, it takes 1.16 times for the subspace iteration method.

Table 2. Eigenvalues of the three-dimensional building structure with concentrated dampers

Mode Number	Eigenvalues
1	-0.13763 + j 3.08907
2	-0. 13763 - j 3.08907
3	-0.13803 + j 3.09109
4	-0. 13803 - j 3.09109
5	-3.52574 + j 2.20649
6	-3.52574 - j 2.20649
7	-0.24236 + j 4.16556
8	-0.24236 - j 4.16556
9	-1.64294 + j 7.02958
10	-1.64294 - j 7.02958
11	-1.65070 + j 7.03590
12	-1.65070 - j 7.03590

Table 3. CPU time (in seconds) spent for the first twelve eigenvalues of the model

Methods	CPU time (Ratio)
Proposed method	8,335.20 (1.00)
Subspace iteration method	9,644.75 (1.16)



YOUNG'S MODULUS(N/M^2) :	*2.1E+11*
MASS DENSITY(KG/M^3) :	*7,850*
CROSS-SECTION INERTIA(M^4) :	*0.833E-05*
CROSS-SECTION AREA(M^2) :	*0.01*
PROPORTIONAL DAMPING COEFFICIENTS:	$\alpha = -0.1755,\ \beta = 0.02005$
CONCENTRATED DAMPING C(N/M/SEC) :	*12,000*

Fig. 5. (a) Three-dimensional building structure (b) Damping from two-layer oundation

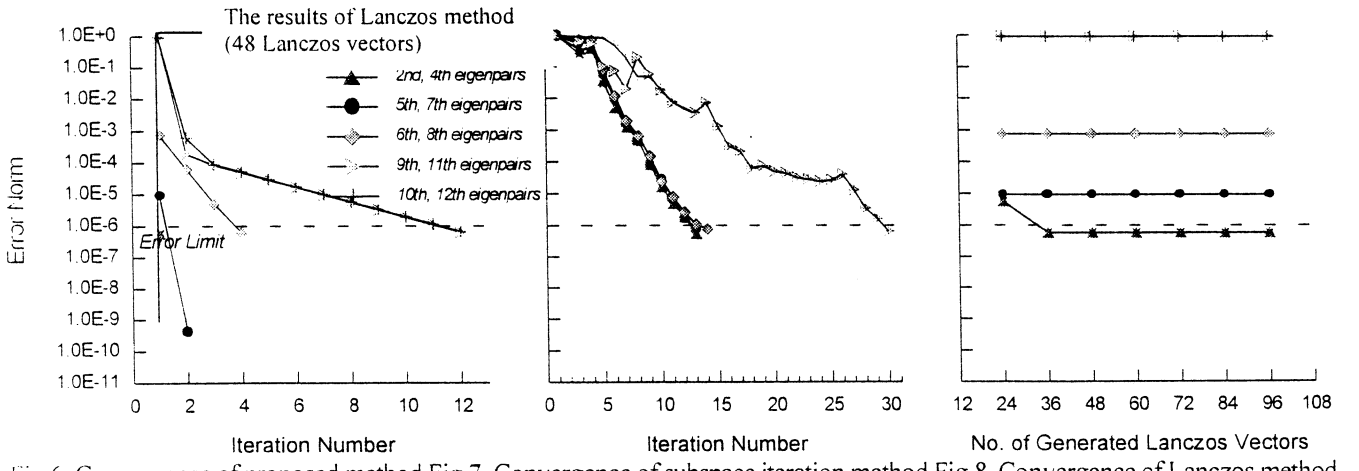


Fig.6. Convergence of proposed method Fig.7. Convergence of subspace iteration method Fig.8. Convergence of Lanczos method