1. INTRODUCTION

To obtain the dynamic response of a large civil structure, it is economic and efficient to superpose the results of a few lowest modes. Therefore, there has been proposed many eigensolution techniques which can find only a set of the lowest modes. The Lanczos and subspace method are belong to this type of technique. In these techniques, however, some important modes can be missed in the calculation process, which may lead to poor results in dynamic analysis. Hence, a checking technique for missed eigenvalues is required in finding the missed one. For the case of undamped system or proportionally damped system, it can be easily found by using the Sturm sequence property [1-4].

However, in the case of the non-proportionally damped systems such as the soil-structure interaction system, the structural control system and composite structures, no counterpart of the Sturm sequence property for undamped systems has been developed yet [5]. Hence, when some important modes are missed for those systems, it may leads to poor results in dynamic analysis.
A number of researchers [6,7] have been performed to solve the eigenproblem with the damping matrix, whereas there have been few studies on a technique to calculate the number of eigenvalues in this case in the literature. Jung et al. [8] proposed a technique of checking missed eigenvalues for eigenproblem with damping matrix using argument principle. This method requires a selection of checking points and the \( LDL^T \) factorization of the characteristic polynomial at those points. The accuracy of the method increases with the number of checking points, so it need more factorization processes to get more exact results.

In this paper, Gleyse's theorem [9], which can count the number of zeros of a characteristic polynomial inside an open unit disk, is used to calculate the number of eigenvalues for eigenproblem with the damping matrix. The characteristic polynomial of an eigenvalue problem is determined by using Chen's algorithm [10] which is considered as both stable and effective. The determinants(minors) of the leading principal submatrices of order \( i \) in the Schur-Cohn matrix can be easily calculated by the \( LDL^T \) factorization process and the final result obtained is very similar to the Sturm sequence property for undamped systems.

This paper is organized as follows. The modified Sturm sequence property is presented and discussed in Chapter 2. In Chapter 3, a numerical example is analyzed to verify the effectiveness of the proposed method. Finally, the concluding remarks are expressed in Chapter 4.

2. MODIFIED STURM SEQUENCE PROPERTY FOR DAMPED SYSTEMS

2.1. The equations of motion of damped systems

In the analysis of dynamic response of structural system, the equation of motion of damped systems can be written as:

\[
M \ddot{u}(t) + C \dot{u}(t) + Ku(t) = 0,
\]

where \( M, K \) and \( C \) are the \( (n \times n) \) mass, stiffness and nonclassical damping matrices, respectively, and \( \ddot{u}(t), \dot{u}(t) \) and \( u(t) \) are the \( (n \times 1) \) acceleration, velocity and displacement vectors, respectively. To find the solution of the free vibration of the system, we consider the following quadratic eigenproblem:

\[
\lambda^2 M \phi + \lambda C \phi + K \phi = 0,
\]

in which \( \lambda \) and \( \phi \) are the eigenvalue and eigenvector of the system. There are \( 2n \) eigenvalues for the system with \( n \) degrees of freedom and these occur either in real pairs or in complex conjugate pairs, depending upon whether they correspond to overdamped or undamped modes.

The common practice is to reformulate the quadratic system of equation to a linear one by doubling the order of the system [6,7] such as:

\[
\begin{bmatrix}
-K & 0 \\
0 & M
\end{bmatrix}
\begin{bmatrix}
\phi \\
\lambda \phi
\end{bmatrix} = \lambda
\begin{bmatrix}
C & M \\
M & 0
\end{bmatrix}
\begin{bmatrix}
\phi \\
\lambda \phi
\end{bmatrix}
\]

In general, \( M \) and \( C \) nonsingular, that is, \( \text{det}(M) \neq 0 \) and \( \text{det}(C) \neq 0 \), so the above equation can
be changed to the form of a standard eigenproblem:

\[ A\psi = \lambda \psi, \]  

(4)

where

\[ A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \quad \psi = \begin{bmatrix} \phi \\ \lambda \phi \end{bmatrix}. \]  

(5)

Observing the above Equation (5), when the mass matrix \( M \) is lumped or banded, the change to the standard eigenproblem can be accomplished without much increase in computing time. The characteristic polynomial of Equation (4) can be represented as:

\[ P(\lambda) = \det(A - \lambda I) = a_n \lambda^{2n} + a_{2n-1} \lambda^{2n-1} + \cdots + a_1 \lambda + a_0 = \sum_{i=0}^{2n} a_i \lambda^i = 0, \]  

(6)

where \( \lambda \) is a complex value and \( a_i (i = 0, 1, \ldots, 2n) \) are the real coefficients.

2.2. The coefficient of the characteristic polynomial

Chen [10] suggested a stable algorithm to obtain the coefficients of the characteristic polynomial of a real square matrix. According to his algorithm some given matrix \( A \):

\[ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,2n-1} & a_{1,2n} \\ a_{21} & a_{22} & \cdots & a_{2,2n-1} & a_{2,2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2n-1,1} & a_{2n-1,2} & \cdots & a_{2n-1,2n-1} & a_{2n-1,2n} \\ a_{2n,1} & a_{2n,2} & \cdots & a_{2n,2n-1} & a_{2n,2n} \end{bmatrix}, \]  

(7)

can be transformed to \( \widetilde{A} \):

\[ \widetilde{A} = \begin{bmatrix} -a_{2n-1} & -a_{2n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \]  

(8)

by applying a sequence of Gauss-elimination like similarity transformations. When some numerical instability occurs during the transformations, modified algorithm by Chen [10] can be used. Since \( \widetilde{A} \) was obtained by applying similar transformations to \( A \), the eigenvalues and eigenvectors of both \( \widetilde{A} \) and \( A \) are same. The characteristic polynomial of \( A \), \( P(\lambda) = \det(A - \lambda I) \) can be obtained by observing the transformed matrix \( \widetilde{A} \), and the characteristic polynomial is:

\[ P(\lambda) = \det(A - \lambda I) = \det(\widetilde{A} - \lambda I) = \lambda^{2n} + a_{2n-1} \lambda^{2n-1} + \cdots + a_1 \lambda + a_0 = 0, \quad (a_{2n} = 1). \]  

(9)

2.3. The number of eigenvalues in an open unit disk.

Gleyse [9] suggested a method of checking the number of eigenvalues of a real polynomial inside an open unit disk by a determinant representation.

Let \( P(\lambda) = \sum_{k=0}^{2n} a_k \lambda^k = 0 \) (\( a_k \) is a real number) then the number of eigenvalues inside the open unit disk can be determined as:

\[ N_k = 2n - S[1, a_1, a_2, \ldots, a_{2n}] \]  

(10)
where $N_i$ is the number of eigenvalues in an open unit disk, $2n$ is the degree of the polynomial, $S[k_0, k_1, k_2, \ldots, k_{2n}]$ is the number of sign changes in the sequence $k_i (i = 0, 1, \ldots, 2n)$ and $d_i (i = 1, 2, \ldots, 2n)$ is the determinants (minors) of the leading principal submatrices of order $i$ in the Schur-Cohn matrix $T$:

$$T = \begin{bmatrix}
\cdots & t_{ij} \\
\cdots & \vdots
\end{bmatrix}, 

\frac{t_{ij}}{a_{ij}} = \frac{\text{det}(\frac{\partial}{\partial a_{\lambda_{i+1} \cdots \lambda_{2n}}} G_{\lambda_1 \cdots \lambda_i}))}{a_{\lambda_{i+1} \cdots \lambda_{2n}} - a_{\lambda_{i+1} \cdots \lambda_{2n}}}.$$

(11)

The process of calculating the number of eigenvalues using the above theorem requires the calculation of the characteristic polynomial of a given matrix, the construction of the Schur-Cohn matrix $T$ and the calculation of the determinants (minors) of the leading principal submatrices of order $i$ in the Schur-Cohn matrix $T$. The coefficients of the characteristic polynomial of a given matrix can be determined by applying previous Chen’s algorithm, and by using these coefficients, each element of the Schur-Cohn matrix can be obtained using Equation (11). The determinants (minors) of the leading principal submatrices of order $i$ in the Schur-Cohn matrix $T$ can be easily determined by applying $LDL^T$ factorization of $T$, which is described in the following section.

2.4. Modified Sturm sequence property.

Gleyse’s theorem [9] considers only about the number of eigenvalues in an open unit disk. To apply his theorem for an open disks of arbitrary radius $\rho$, substitute $\lambda = \rho \lambda$ (\rho is an real number) to Equation (6), then:

$$P(\lambda) = a_{2n} \lambda^{2n} + a_{2n-1} \rho^{2n-1} \lambda^{2n-1} + \cdots + a_{1} \rho \lambda + a_{0}$$

$$= a_{2n} \lambda^{2n} + a_{2n-1} \lambda^{2n-1} + \cdots + a_{1} \lambda + \bar{a} = \sum_{i=0}^{2n} \bar{a}_{i} \lambda^{i} = 0,$$

(12)

where $\bar{a}_i = a_i \rho^i (i = 0, 1, \ldots, 2n)$ are modified coefficients.

Using the modified coefficients $\bar{a}_i (i = 0, 1, \ldots, 2n)$ in Equation (12), His theorem can be extended to calculate the number of eigenvalues in the open disks of arbitrary radius $\rho$. The calculation of $d_i (i = 1, 2, \ldots, 2n)$ can be easily performed by the $LDL^T$ factorization of the Schur-Cohn matrix $T$. If $T = LDL^T$, then:

$$T_i = L_i D_i L_i^T$$

(13)

where $T_i$ is the leading principal submatrices of order $i$ in the $T$, $L_i$ is the leading principal submatrices of order $i$ in the $L$ and $D_i$ is the leading principal submatrices of order $i$ in the $D$.

Therefore each $d_i (i = 1, 2, \ldots, 2n)$ can be easily obtained as:

$$d_i = \det(T_i) = \det(L_i D_i L_i^T) = \det(D_i)$$

$$= d_{i1} \times d_{i2} \times \cdots \times d_{ii} = \prod_{k=1}^{i} d_{kk}$$

(14)

Considering Equation (10), we only need to know the signs of each $d_i$ because the unknown
value of $S[1, d_1, d_2, \ldots, d_{2n}]$ depends only on sign changes of each $d_i (i=1, 2, \ldots, 2n)$ and from Equation (14) the signs of each $d_i$ can be determined from the number of negative elements of each diagonal elements of $D_0$, which is very similar to Strum sequence property for undamped systems.

3. NUMERICAL EXAMPLES

To show the effectiveness of the proposed method, the plane frame structure with lumped dampers which has multiple eigenvalues is considered.

3.1. Plane frame structure with lumped dampers

In this example, a plane frame structure with lumped dampers is presented. The geometric configuration and material properties are shown in Figure 1. The model is discretized in 12 beam elements resulting in the system of dynamic equation with a total of 18 degrees of freedom. Thus, the order of the associated eigenproblem is 36. The consistent mass matrix is derived from the classical damping given by $C = \alpha K + \beta M$ and concentrated dampers. All the eigenvalues are calculated by the Lanczos method developed by Kim and Lee [7] and their radius from the origin in the complex plane are calculated by $\rho_i = |\lambda_i|$ as in Table I. Two first two cases are for checking in between eigenvalues. The third is for checking all the eigenvalues of the system. The radius of the open disk is determined by $\rho = 1.005|\lambda|$. 

![Figure 1. Plane frame structure with lumped dampers.](image)

For each case calculated coefficient of the characteristic polynomial $\overline{\alpha}$ and sign of $d_i$ are as in Table I. Using the sign of $d_i$ Table I, the number of eigenvalues for each cases are calculated as followings:

Case 1: $\rho = 1.005|\lambda| = 81.555$

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\[ N_\lambda = 2n - S[l_1, d_1, d_2, \ldots, d_{2n}] \]
\[ N_\lambda = 36 - 24 = 12 \]

**Case 2:**  \[ \rho = 1.005|\lambda_{24}| = 203.681 \]

\[ N_\lambda = 2n - S[l_1, d_1, d_2, \ldots, d_{2n}] \]

### Table 1. The calculated eigenvalues, characteristic coefficient \( \overline{a_i} \) and sign of \( d_i \)

| \( \lambda \) | Real | Imaginary | \( \rho = 1.005|\lambda_{12}| = 81.555 \) | \( \rho = 1.005|\lambda_{24}| = 203.68 \) | \( \rho = 1.005|\lambda_{32}| = 219.12 \) |
|---|---|---|---|---|---|
| 0 | -1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 1 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 2 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 3 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 4 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 5 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 6 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 7 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 8 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 9 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 10 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 11 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 12 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 13 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 14 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 15 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 16 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 17 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 18 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 19 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 20 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 21 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 22 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 23 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 24 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 25 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 26 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 27 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 28 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 29 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 30 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 31 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 32 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 33 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 34 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 35 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |
| 36 | 1.1369 | -46.2187 | 46.2327 | 5.606e+12 | 1.822e+09 | 8.88e+15 |

* \( \text{sign}(d_i) \) represents the sign of \( d_i \), so \( \text{sign}(d_i) \) is '+1' when \( d_i > 0 \) and \( \text{sign}(d_i) \) is '-1' when \( d_i < 0 \).

\( d_0 \) is defined as equal to 1.
\[ N_\lambda = 36 - 4 = 32 \]

Case 4: \[ \rho = 1.005|\lambda_{36}| = 219.123 \]
\[ N_\lambda = 2n - S\{d_1, d_2, \ldots, d_{2n}\} \]
\[ N_\lambda = 36 - 0 = 36 \]

Referring to Table I, the number of eigenvalues which is inside open disks of radius \( \rho = 1.005|\lambda_{32}| = 81.555 \), \( \rho = 1.005|\lambda_{20}| = 203.681 \) and \( \rho = 1.005|\lambda_{36}| = 219.123 \) are 12, 32 and 36 which are exactly agree with the calculated values. As seen from this result. Therefore, we verify that the proposed method can exactly check the number of eigenvalues inside some open disk of arbitrary radius.

4. CONCLUSIONS

A technique of calculating the number of eigenvalues inside an open disk of arbitrary radius was given. The technique is based on Chens algorithm and Gleyse's theorem and can be used to check the missed eigenvalues for the eigenproblem with damping matrix. By analyzing the numerical examples, it is verified that the proposed method can exactly check the number of eigenvalues for distinct or multiple eigenvalues for damped systems.

The technique by Jung et al. should find the variation of arguments of complex numbers along a predefined path. Therefore, a large number of checking points should be used to obtain accurate result. However, the proposed method can exactly find the number of eigenvalues by performing the factorization process only once. In result, much effort in finding the number of eigenvalue of larger structures with damping matrix can be eliminated by the proposed method.

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