감쇠시스템을 위한 개선된 Sturm 수열 성질
Modified Sturm Sequence Property for Damped Systems

조지성* 김병완** 이인원****
Jiseong Jo Byungwan Kim In-won Lee

요 지

비감쇠 혹은 비비례감쇠 시스템의 고유치를 구하기 위한 대부분의 방법들은 저차의 몇 개의 모드만을 사용하여 동적응답을 구하는 경우 누락된 고유치의 존재여부를 검사하기 위해 잘 알려진 Sturm 수열 성질(Sturm sequence property)을 이용한다. 반면 감쇠시스템 즉, 지반-구조물 상호작용 시스템, 구조물의 진동제어 시스템, 복합재료 구조물과 같은 경우에는 저차 몇 개의 모드만을 사용할 경우 누락 고유치를 검사할 수 있는 효율적인 기법이 아직 확립되지 않은 상태이다. 본 논문에서는 Gleyse의 정리를 이용하여 감쇠스템의 누락된 고유치를 검사하는 기법을 제안하고 이 방법의 효용성을 수치예제를 통하여 검증하였다.

1. INTRODUCTION

To obtain the dynamic response of a large civil structure, it is economic and efficient to superpose the results of a few lowest modes. Therefore, there has been proposed many eigensolution techniques which can find only a set of the lowest modes. The Lanczos and subspace method are belong to this type of technique. In these techniques, however, some important modes can be missed in the calculation process, which may lead to poor results in dynamic analysis. Hence, a checking technique for missed eigenvalues is required in finding the missed one. For the case of undamped system or proportionally damped system, it can be easily found by using the Sturm sequence property [1-4].

However, in the case of the non-proportionally damped systems such as the soil-structure interaction system, the structural control system and composite structures, no counterpart of the Sturm sequence property for undamped systems has been developed yet [5]. Hence, when some important modes are missed for those systems, it may leads to poor results in dynamic analysis.

* 한국과학기술원 토목공학과 박사과정
** 한국과학기술원 토목공학과 박사과정
** 한국과학기술원 토목공학과 교수, 정회원

A number of researchers [6,7] have been performed to solve the eigenproblem with the damping matrix, whereas there have been few studies on a technique to calculate the number of eigenvalues in this case in the literature. Jung et al. [8] proposed a technique of checking missed eigenvalues for eigenproblem with damping matrix using argument principle. This method requires a selection of checking points and the LDL^T factorization of the characteristic polynomial at those points. The accuracy of the method increases with the number of checking points, so it need more factorization processes to get more exact results.

In this paper, Gleyse's theorem [9], which can count the number of zeros of a characteristic polynomial inside an open unit disk, is used to calculate the number of eigenvalues for eigenproblem with the damping matrix. The characteristic polynomial of an eigenvalue problem is determined by using Chen's algorithm [10] which is considered as both stable and effective. The determinants(minors) of the leading principal submatrices of order i in the Schur-Cohn matrix can be easily calculated by the LDL^T factorization process and the final result obtained is very similar to the Sturm sequence property for undamped systems.

This paper is organized as follows. The modified Sturm sequence property is presented and discussed in Chapter 2. In Chapter 3, a numerical example is analyzed to verify the effectiveness of the proposed method. Finally, the concluding remarks are expressed in Chapter 4.

2. MODIFIED STURM SEQUENCE PROPERTY FOR DAMPED SYSTEMS

2.1. The equations of motion of damped systems

In the analysis of dynamic response of structural system, the equation of motion of damped systems can be written as:

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = 0, \tag{1}$$

where M, K and C are the $(n \times n)$ mass, stiffness and nonclassical damping matrices, respectively, and $\ddot{u}(t)$, $\dot{u}(t)$ and $u(t)$ are the $(n \times 1)$ acceleration, velocity and displacement vectors, respectively. To find the solution of the free vibration of the system, we consider the following quadratic eigenproblem:

$$\lambda^2 M\phi + \lambda C\phi + K\phi = 0, \tag{2}$$

in which λ and ϕ are the eigenvalue and eigenvector of the system. There are $2n$ eigenvalues for the system with n degrees of freedom and these occur either in real pairs or in complex conjugate pairs, depending upon whether they correspond to overdamped or undamped modes.

The common practice is to reformulate the quadratic system of equation to a linear one by doubling the order of the system [6,7] such as:

$$\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix} = \lambda \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix} \tag{3}$$

In general, M and C nonsingular, that is, $det(M) \neq 0$ and $det(C) \neq 0$, so the above equation can be changed to the form of a standard eigenproblem:

$$A\psi = \lambda\psi, \tag{4}$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \psi = \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix}. \tag{5}$$

Observing the above Equation (5), when the mass matrix M is lumped or banded, the change to the standard eigenproblem can be accomplished without much increase in computing time. The characteristic polynomial of Equation (4) can be represented as:

$$P(\lambda) = \det(A - \lambda I) = a_{2n}\lambda^{2n} + a_{2n-1}\lambda^{2n-1} + \cdots + a_1\lambda + a_0 = \sum_{i=0}^{2n} a_i\lambda^i = 0, \tag{6}$$

where λ is a complex value and $a_i (i=0, 1, \cdots, 2n)$ are the real coefficients.

2.2. The coefficient of the characteristic polynomial

Chen [10] suggested a stable algorithm to obtain the coefficients of the characteristic polynomial of a real square matrix. According to his algorithm some given matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,2n-1} & a_{1,2n} \\ a_{21} & a_{22} & \cdots & a_{2,2n-1} & a_{2,2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2n-1,1} & a_{2n-1,2} & \cdots & a_{2n-1,2n-1} & a_{2n-1,2n} \\ a_{2n,1} & a_{2n,2} & \cdots & a_{2n,2n-1} & a2_{n,2n} \end{bmatrix}, \tag{7}$$

can be transformed to $\overline{A}$:

$$\overline{A} = \begin{bmatrix} -a_{2n-1} & -a_{2n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{8}$$

by applying a sequence of Gauss-elimination like similarity transformations. When some numerical instability occurs during the transformations, modified algorithm by Chen [10] can be used. Since $\overline{A}$ was obtained by applying similar transformations to A, the eigenvalues and eigenvectors of both $\overline{A}$ and A are same. The characteristic polynomial of A, $P(\lambda) = det(A - \lambda I)$ can be obtained by observing the transformed matrix $\overline{A}$, and the characteristic polynomial is:

$$P(\lambda) = \det(A - \lambda I) = \det(\overline{A} - \lambda I) = \lambda^{2n} + a_{2n-1}\lambda^{2n-1} + \cdots + a_1\lambda + a_0 = 0, (a_{2n} = 1). \tag{9}$$

2.3. The number of eigenvalues in an open unit disk.

Gleyse [9] suggested a method of checking the number of eigenvalues of a real polynomial inside an open unit disk by a determinant representation.

Let $P(\lambda) = \sum_{h=0}^{2n} a_h \lambda_h = 0$ (a_h is a real number) then the number of eigenvalues inside the open unit disk can be determined as:

$$N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n} \tag{10}$$

where N_λ is the number of eigenvalues in an open unit disk, 2n is the degree of the polynomial, $S[k_0, k_1, k_2, \cdots, k_{2n}]$ is the number of sign changes in the sequence $k_i(i=0,1,\cdots,2n)$ and $d_i(i=1,2,\cdots,2n)$ is the determinants(minors) of the leading principal submatrices of order i in the Schur-Cohn matrix T:

$$T = \begin{bmatrix} \ddots & & \\ & t_{ij} & \\ & & \ddots \end{bmatrix}, \; t_{ij} = \sum_{h=0}^{\min(i,j)} (a_{2n-i+h} a_{2n-j+h} - a_{i-h} a_{j-h} \quad (11)$$

The process of calculating the number of eigenvalues using the above theorem requires the calculation of the characteristic polynomial of a given matrix, the construction of the Schur-Cohn matrix T and the calculation of the determinants (minors) of the leading principal submatrices of order i in the Schur-Cohn matrix T. The coefficients of the characteristic polynomial of a given matrix can be determined by applying previous Chens algorithm, and by using these coefficients, each elements of the Schur-Cohn matrix can be obtained using Equation (11). The determinants(minors) of the leading principal submatrices of order i in the Schur-Cohn matrix T can be easily determined by applying LDL^T factorization of T, which is described in the following section.

2.4. Modified Sturm sequence property.

Gleyse's theorem [9] considers only about the number of eigenvalues in an open unit disk. To apply his theorem for an open disks of arbitrary radius ρ, substitute $\lambda = \rho\bar{\lambda}$(ρ is an real number) to Equation (6), then:

$$\begin{aligned} P(\bar{\lambda}) &= a_{2n}\rho^{2n}\bar{\lambda}^{2n} + a_{2n-1}\rho^{2n-1}\bar{\lambda}^{2n-1} + \cdots + a_1\rho\bar{\lambda} + a_0 \\ &= \bar{a}_{2n}\bar{\lambda}^{2n} + \bar{a}_{2n-1}\bar{\lambda}^{2n-1} + \cdots + \bar{a}_1\bar{\lambda} + \bar{a} = \sum_{i=0}^{2n} \bar{a}_i\bar{\lambda}^i = 0, \end{aligned} \quad (12)$$

where $\overline{a_i} = a_i\rho^i(i=0,1,\cdots,2n)$are modified coefficients.

Using the modified coefficients $\overline{a_i}(i=0,1,\cdots,2n)$ in Equation (12), His theorem can be extended to calculate the number of eigenvalues in the open disks of arbitrary radius ρ. The calculation of $d_i(i=1,2,\cdots,2n)$ can be easily performed by the LDL^T factorization of the Schur-Cohn matrix T. If $T=LDL^T$, then:

$$T_i = L_i D_i L_i^T \quad (13)$$

where T_i is the leading principal submatrices of order i in the T, L_i is the leading principal submatrices of order i in the L and D_i is the leading principal submatrices of order i in the D. Therefore each $d_i(i=1,2,\cdots,2n)$ can be easily obtained as:

$$\begin{aligned} d_i &= \det(T_i) = \det(L_i D_i L_i^T) = \det(D_i) \\ &= d_{11} \times d_{22} \times \cdots \times d_{ii} = \prod_{h=1}^{i} d_{hh} \end{aligned} \quad (14)$$

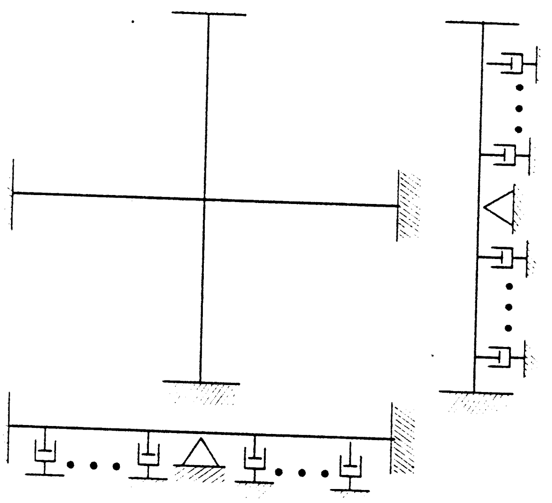
Considering Equation (10), we only need to know the signs of each d_i because the unknown value of $S[1, d_1, d_2, \cdots, d_{2n}]$ depends only on sign changes of each $d_i(i=1,2,\cdots,2n)$ and from Equation (14) the signs of each d_i can be determined from the number of negative elements of each diagonal elements of D_i, which is very similar to Strum sequence property for undamped systems.

3. NUMERICAL EXAMPLES

To show the effectiveness of the proposed method, the plane frame structure with lumped dampers which has multiple eigenvalues is considered.

3.1. Plane frame structure with lumped dampers

In this example, a plane frame structure with lumped dampers is presented. The geometric configuration and material properties are shown in Figure 1. The model is discretized in 12 beam elements resulting in the system of dynamic equation with a total of 18 degrees of freedom. Thus, the order of the associated eigenproblem is 36. The consistent mass matrix is derived from the classical damping given by $C=\alpha K+\beta M$ and concentrated dampers. All the eigenvalues are calculated by the Lanczos method developed by Kim and Lee [7] and their radius from the origin in the complex plane are calculated by $\rho_i=|\lambda_i|$ as in Table I. Two First two cases are for checking in between eigenvalues. The third is for checking all the eigenvalues of the system. The radius of the open disk is determined by $\rho=1.005|\lambda|$.



Youngs Modulus: 1000
Mass Density: 1.0
Cross-section Inertia: 1.0
Cross-section Area: 1.0
Span Length: 6.0
Concentrated Damping: 0.3
Rayleigh Damping Coeff.:
$\alpha=0.001, \beta=0.001$

Figure 1. Plane frame structure with lumped dampers.

For each case calculated coefficient of the characteristic polynomial $\overline{a}_i$ and sign of d_i are as in Table I. Using the sign of d_i Table I, the number of eigenvalues for each cases are calculated as followings:

Case 1: $\rho=1.005|\lambda_{12}|=81.555$

$$N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}$$

$$N_\lambda = 36 - 24 = 12$$

Case 2: $\rho = 1.005|\lambda_{32}| = 203.681$

$$N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}$$

Table I. The calculated eigenvalues, characteric coefficient $\overline{a_i}$ and sign of d_i

i	Eigenvalues(λ)		Radius	$\rho = 1.005\lvert\lambda_{12}\rvert = 81.555$		$\rho = 1.005\lvert\lambda_{32}\rvert = 203.68$		$\rho = 1.005\lvert\lambda_{36}\rvert = 219.12$	
	Real	Imaginary	($\rho = \lvert\lambda\rvert$)	$\overline{a_i}$	$sign(d_i)$	$\overline{a_i}$	$sign(d_i)$	$\overline{a_i}$	$sign(d_i)$
0	-	-	-	3.002e+11	+	1.000e+05	+	3.002e+11	+
1	-1.1369	-46.2187	46.2327	4.492e+11	-	3.737e+05	+	1.207e+12	+
2	-1.1369	-46.2187	46.2327	5.594e+12	+	1.162e+07	+	4.039e+13	+
3	-1.1369	-46.2187	46.2327	7.588e+12	-	3.937e+07	+	1.472e+14	+
4	-1.1369	-46.2187	46.2327	4.588e+13	+	5.946e+08	+	2.391e+15	+
5	-1.3731	-51.1333	51.1517	5.630e+13	-	1.822e+09	+	7.883e+15	+
6	-1.3731	-51.1333	51.1517	2.193e+14	+	1.772e+10	+	8.249e+16	+
7	-1.3731	-51.1333	51.1517	2.428e+14	-	4.903e+10	+	2.455e+17	+
8	-1.3731	-51.1333	51.1517	6.826e+14	+	3.441e+11	+	1.854e+18	+
9	-3.3902	-81.0872	81.1490	6.804e+14	-	8.567e+11	+	4.965e+18	+
10	-3.3902	-81.0872	81.1490	1.468e+15	+	4.615e+12	+	2.877e+19	+
11	-3.3902	-81.0872	81.1490	1.313e+15	-	1.031e+13	+	6.914e+19	+
12	-3.3902	-81.0872	81.1490	2.258e+15	+	4.429e+13	+	3.196e+20	+
13	-3.9407	-87.4771	87.5659	1.806e+15	-	8.847e+13	+	6.868e+20	+
14	-3.9407	-87.4771	87.5659	2.544e+15	+	3.112e+14	+	2.599e+21	+
15	-3.9407	-87.4771	87.5659	1.812e+15	-	5.537e+14	+	4.975e+21	+
16	-3.9407	-87.4771	87.5659	2.128e+15	-	1.624e+15	+	1.570e+22	+
17	-8.1642	-127.4394	127.7006	1.345e+15	+	2.563e+15	+	2.665e+22	+
18	-8.1642	-127.4394	127.7006	1.335e+15	-	6.353e+15	+	7.106e+22	+
19	-8.1642	-127.4394	127.7006	7.439e+14	-	8.843e+15	+	1.064e+23	+
20	-8.1642	-127.4394	127.7006	6.297e+14	-	1.870e+16	+	2.420e+23	+
21	-10.2629	-142.8367	143.2049	3.074e+14	+	2.279e+16	+	3.174e+23	+
22	-10.2629	-142.8367	143.2049	2.234e+14	+	4.137e+16	+	6.198e+23	+
23	-10.2629	-142.8367	143.2049	9.457e+13	-	4.374e+16	+	7.051e+23	+
24	-10.2629	-142.8367	143.2049	5.923e+13	-	6.841e+16	+	1.186e+24	+
25	-14.8662	-171.7301	172.3720	2.146e+13	+	6.189e+16	+	1.155e+24	+
26	-14.8662	-171.7301	172.3720	1.159e+13	-	8.350e+16	+	1.676e+24	+
27	-14.8662	-171.7301	172.3720	3.522e+12	-	6.337e+16	+	1.368e+24	+
28	-14.8662	-171.7301	172.3720	1.639e+12	-	7.364e+16	+	1.711e+24	+
29	-20.5387	-201.6249	202.6683	4.050e+11	-	4.545e+16	-	1.136e+24	+
30	-20.5387	-201.6249	202.6683	1.616e+11	+	4.528e+16	+	1.217e+24	+
31	-20.5387	-201.6249	202.6683	3.080e+10	-	2.156e+16	-	6.236e+23	+
32	-20.5387	-201.6249	202.6683	1.044e+10	-	1.826e+16	-	5.682e+23	+
33	-23.7699	-216.7332	218.0328	1.386e+09	+	6.051e+15	-	2.026e+23	+
34	-23.7699	-216.7332	218.0328	3.940e+08	+	4.296e+15	+	1.547e+23	+
35	-23.7699	-216.7332	218.0328	2.783e+07	+	7.579e+14	+	2.937e+22	+
36	-23.7699	-216.7332	218.0328	6.490e+06	+	4.414e+14	+	1.840e+22	+

* $sign(d_i)$ represents the sign of d_i, so $sign(d_i)$ is '+' when $d_i > 0$ and $sign(d_i)$ is '-' when $d_i < 0$. d_0 is defined as equal to 1.

$$N_\lambda = 36 - 4 = 32$$

Case 4: $\rho = 1.005|\lambda_{36}| = 219.123$

$$N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}$$

$$N_\lambda = 36 - 0 = 36$$

Referring to Table I, the number of eigenvalues which is inside open disks of radius $\rho = 1.005|\lambda_{12}| = 81.555$, $\rho = 1.005|\lambda_{32}| = 203.681$ and $\rho = 1.005|\lambda_{36}| = 219.123$ are 12, 32 and 36 which are exactly agree with the calculated values. As seen from this result. Therefore, we verify that the proposed method can exactly check the number of eigenvalues inside some open disk of arbitrary radius.

4. CONCLUSIONS

A technique of calculating the number of eigenvalues inside an open disk of arbitrary radius was given. The technique is based on Chens algorithm and Gleyse's theorem and can be used to check the missed eiegnvalues for the eigenproblem with damping matrix. By analyzing the numerical examples, it is verified that the proposed method can exactly check the number of eigenvalues for distinct or multiple eigenvalues for damped systems.

The technique by Jung et al. should find the variation of arguments of complex numbers along a predefined path. Therefore, a large number of checking points sholud be used to obtain accurate result. However, the proposed method can exactly find the number of eigenvalues by performing the factorization process only once. In result, much effort in finding the number of eigenvalue of larger structures with damping matrix can be eliminated by the proposed method.

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REFERENCES

1. Bathe KJ. *Finite Element Procedures*. Prentice-Hall, Inc., 1996.
2. Meirovitch L. *Computational Methods in Structural Dynamics*. Sijthoff & Noordhoff, 1980.
3. Petyt M. *Introduction to Finite Element Vibration Analysis*. Cambridge University Press, 1990.
4. Hughes TJR. *The Finite Element Method; Linear Static and Dynamic Finite Element Analysis*. Prentice-Hall, Inc., 1987.
5. Newland DE. *Mechanical Vibration Analysis and Computation*. Longman Scientific &

Technical, 1989.

6. Rajakumar C. Lanczos algorithm for the quadratic eigenvalue problem in engineering applications. *International Journal for Numerical Methods in Engineering* 1993; **105:** 1-22.
7. Kim MC, Lee IW. Solution Eigenproblems for Non-Proportional Damping System by Lanczos Method. *Earthquake Engineering and Structural Dynamics* 1999; **28:** 157-172
8. Jung HJ, Kim DH and Lee IW. Technique of Checking Missed Eigenvalues for Eigenproblem with Damping Matrix. *International Journal for Numerical Methods in Engineering* 2001; **50:** 55-66.
9. Gleyse B, Moflih M. Exact Computation of the Number of Zeros of a Real Polynomial in the Open Unit Disk by a Determinant Representation. *Computers and Mathematics with Applications.* 1999; **38**: 257-263.
10. Wang J, Chen CT. On the Computation of the Characteristic Polynomial of a Matrix. *IEEE Transaction on Automatic Control* 1982; **27:** 449-451.
11. Chen HC, Taylor RL. Solution of Eigenproblems for Damped Structural Systems by Lanczos Algorithm. *Computers and Structures* 1988; **30:** 151-161.