

감쇠시스템을 위한 개선된 Sturm 수열 성질

Modified Sturm Sequence Property for Damped Systems

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요약: 비감쇠 혹은 비비례감쇠 시스템의 고유치를 구하기 위한 대부분의 방법들은 저차의 몇 개의 모드만을 사용하여 동적응답을 구하는 경우 누락된 고유치의 존재여부를 검사하기 위해 잘 알려진 Sturm 수열 성질(Sturm sequence property)을 이용한다. 반면 감쇠시스템 즉, 지반-구조물 상호작용 시스템, 구조물의 진동제어 시스템, 복합재료 구조물과 같은 경우에는 저차 몇 개의 모드만을 사용할 경우 누락 고유치를 검사할 수 있는 효율적인 기법이 아직 확립되지 않은 상태이다. 본 논문에서는 Gleyse의 정리를 이용하여 감쇠시스템의 누락된 고유치를 검사하는 기법을 제안하고 이 방법의 효용성을 수치예제를 통하여 검증하였다.

Abstract: Most of the eigenvalue analysis methods for the undamped or proportionally damped systems use the well-known Sturm sequence property to check the missed eigenvalues when only a set of the lowest modes is to be used for large structures. However, in the case of the non-proportionally damped systems such as the soil-structure interaction system, the structural control system and the composite structures, no counterpart of the Sturm sequence property for undamped systems has been developed yet. Hence, when some important modes are missed for those systems, it may leads to poor results in dynamic analysis. In this paper, a technique for calculating the number of eigenvalues inside the open disk of arbitrary radius for the eigenproblem with the damping matrix is proposed by applying Chens algorithm and Gleyse's theorem. To verify the applicability of the proposed method, two numerical examples are considered.

핵심용어: Sturm 수열 성질, 비비례감쇠, 감쇠를 포함한 고유치문제, 특성다항식, 누락된 고유치, Schur-Cohn 행렬

Keywords: Sturm sequence property, non-proportional damping, eigenproblem with damping, characteristic polynomial, missed eigenvalues, Schur-Cohn matrix.

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1. INTRODUCTION

To obtain the dynamic response of a large civil structure, it is economic and efficient to superpose the results of a few lowest modes. Therefore, there has been proposed many eigensolution techniques which can find only a set of the lowest modes. The Lanczos and subspace method are belong to this type of technique. In these techniques, however, some important modes can be missed in the calculation process, which may lead to poor results in dynamic analysis. Hence, a checking technique for missed eigenvalues is required in finding the missed one. For the case of undamped system or proportionally damped system, it can be easily found by using the Sturm sequence property [1-4].

However, in the case of the non-proportionally damped systems such as the soil-structure interaction system, the structural control system and composite structures, no counterpart of the Sturm sequence property for undamped systems has been developed yet [5]. Hence, when some important modes are missed for those systems, it may leads to poor results in dynamic analysis.

In this paper, Gleyse's theorem [9], which can count the number of zeros of a characteristic polynomial inside an open unit disk, is used to calculate the number of eigenvalues for eigenproblem with the damping matrix. The characteristic polynomial of an eigenvalue problem is determined by using Chen's algorithm [10] which is considered as both stable and effective. The determinants(minors) of the leading principal submatrices of order i in the Schur-Cohn matrix can be easily calculated by the LDL^T factorization process and the final result

obtained is very similar to the Sturm sequence property for undamped systems.

2. MODIFIED STURM SEQUENCE PROPERTY FOR DAMPED SYSTEMS

2.1. The equations of motion of damped systems

In the analysis of dynamic response of structural system, the equation of motion of damped systems can be written as:

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = 0, \quad (1)$$

where M, K and C are the $(n \times n)$ mass, stiffness and nonclassical damping matrices, respectively, and $\ddot{u}(t), \dot{u}(t)$ and $u(t)$ are the $(n \times 1)$ acceleration, velocity and displacement vectors, respectively. To find the solution of the free vibration of the system, we consider the following quadratic eigenproblem:

$$\lambda^2 M \phi + \lambda C \phi + K \phi = 0, \quad (2)$$

in which λ and ϕ are the eigenvalue and eigenvector of the system. There are $2n$ eigenvalues for the system with n degrees of freedom and these occur either in real pairs or in complex conjugate pairs, depending upon whether they correspond to overdamped or undamped modes.

The common practice is to reformulate the quadratic system of equation to a linear one by doubling the order of the system [6,7] such as:

$$\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda \phi \end{Bmatrix} = \lambda \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda \phi \end{Bmatrix} \quad (3)$$

In general, M and C nonsingular, that is, $\det(M) \neq 0$ and $\det(C) \neq 0$, so the above equation can be changed to the form of a standard eigenproblem:

$$A \psi = \lambda \psi, \quad (4)$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \psi = \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix}. \quad (5)$$

Observing the above Equation (5), when the mass matrix M is lumped or banded, the change to the standard eigenproblem can be accomplished without much increase in computing time. The characteristic polynomial of Equation (4) can be represented as:

$$\begin{aligned} P(\lambda) &= \det(A - \lambda I) \\ &= a_{2n}\lambda^{2n} + a_{2n-1}\lambda^{2n-1} + \dots + a_1\lambda + a_0, \\ &= \sum_{i=1}^{2n} a_i \lambda^i = 0, \end{aligned} \quad (6)$$

where λ is a complex value and $a_i (i=0, 1, \dots, 2n)$ are the real coefficients.

2.2. The coefficient of the characteristic polynomial

Chen [10] suggested a stable algorithm to obtain the coefficients of the characteristic polynomial of a real square matrix. According to his algorithm some given matrix A :

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,2n-1} & a_{1,2n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,2n-1} & a_{2,2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2n-1,1} & a_{2n-1,2} & \dots & a_{2n-1,2n-1} & a_{2n-1,2n} \\ a_{2n,1} & a_{2n,2} & \dots & a_{2n,2n-1} & a_{2n,2n} \end{bmatrix}, \quad (7)$$

can be transformed to \bar{A} :

$$\bar{A} = \begin{bmatrix} -a_{2n-1} & -a_{2n-2} & \dots & -a_1 & -a_0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad (8)$$

by applying a sequence of Gauss-elimination like similarity transformations. When some numerical instability occurs during the transformations, modified algorithm by Chen [10] can be used. Since \bar{A} was obtained by applying similar transformations to A , the

eigenvalues and eigenvectors of both \bar{A} and A are same. The characteristic polynomial of A , $P(\lambda) = \det(A - \lambda I)$ can be obtained by observing the transformed matrix \bar{A} , and the characteristic polynomial is:

$$\begin{aligned} P(\lambda) &= \det(A - \lambda I) \\ &= \det(\bar{A} - \lambda I) \\ &= \lambda^{2n} + a_{2n-1}\lambda^{2n-1} + \dots + a_1\lambda + a_0 = 0, (a_{2n} = 1) \end{aligned} \quad (9)$$

2.3. The number of eigenvalues in an open unit disk

Glyse [9] suggested a method of checking the number of eigenvalues of a real polynomial inside an open unit disk by a determinant representation.

Let $P(\lambda) = \sum_{h=0}^{2n} a_h \lambda^h = 0$ (a_h is a real number) then the number of eigenvalues inside the open unit disk can be determined as:

$$N_\lambda = 2n - S[1, d_1, d_2, \dots, d_{2n}] \quad (10)$$

where N_λ is the number of eigenvalues in an open unit disk, $2n$ is the degree of the polynomial, $S[k_0, k_1, k_2, \dots, k_{2n}]$ is the number of sign changes in the sequence $k_i (i=0, 1, \dots, 2n)$ and $d_i (i=1, 2, \dots, 2n)$ is the determinants (minors) of the leading principal submatrices of order i in the Schur-Cohn matrix T :

$$\begin{aligned} T &= \begin{bmatrix} \ddots & & & & \\ & \ddots & & & \\ & & t_{ij} & & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix}, \\ t_{ij} &= \sum_{h=0}^{\min(i,j)} (a_{2n-i+h} a_{2n-j+h} - a_{i-h} a_{j-h}). \end{aligned} \quad (11)$$

2.4. Modified Sturm sequence property.

Glyse's theorem [9] considers only about the number of eigenvalues in an open unit disk. To apply his theorem for an open disks of arbitrary radius ρ , substitute $\lambda = \rho \bar{\lambda}$ (ρ is

an real number) to Equation (6), then:

$$\begin{aligned} P(\bar{\lambda}) &= a_{2n}\rho^{2n}\bar{\lambda}^{2n} + a_{2n-1}\rho^{2n-1}\bar{\lambda}^{2n-1} + \dots + a_1\rho\bar{\lambda} + a_0 \\ &= \bar{a}_{2n}\bar{\lambda}^{2n} + \bar{a}_{2n-1}\bar{\lambda}^{2n-1} + \dots + \bar{a}_1\bar{\lambda} + \bar{a}_0 \\ &= \sum_{i=0}^{2n} \bar{a}_i \bar{\lambda}^i = 0, \end{aligned} \quad (12)$$

where $\bar{a}_i = a_i \rho^i$ ($i=0, 1, \dots, 2n$) are modified coefficients.

Using the modified coefficients \bar{a}_i ($i=0, 1, \dots, 2n$) in Equation (12), His theorem can be extended to calculate the number of eigenvalues in the open disks of arbitrary radius ρ . The calculation of d_i ($i=1, 2, \dots, 2n$) can be easily performed by the LDL^T factorization of the Schur-Cohn matrix T . If $T = LDL^T$, then:

$$T_i = L_i D_i L_i^T \quad (13)$$

where T_i is the leading principal submatrices of order i in the T , L_i is the leading principal submatrices of order i in the L and D_i is the leading principal submatrices of order i in the D . Therefore each d_i ($i=1, 2, \dots, 2n$) can be easily obtained as:

$$\begin{aligned} d_i &= \det(T_i) = \det(L_i D_i L_i^T) = \det(D_i) \\ &= d_{11} \times d_{22} \times \dots \times d_{ii} = \prod_{h=1}^i d_{hh} \end{aligned} \quad (14)$$

Considering Equation (10), we only need to know the signs of each d_i because the unknown value of $S[1, d_1, d_2, \dots, d_{2n}]$ depends only on sign changes of each d_i ($i=1, 2, \dots, 2n$) and from Equation (14) the signs of each d_i can be determined from the number of negative elements of each diagonal elements of D_i , which is very similar to Sturm sequence property for undamped systems.

3. A NUMERICAL EXAMPLE

To show the effectiveness of the proposed method, a simple spring-mass-damper system that has the exact analytical eigenvalues is considered to verify that the proposed method can exactly calculate the number of eigenvalues in the open disk of arbitrary radius for the eigenproblem with the damping matrix.

3.1. Simple Spring-Mass-Damper System⁽¹¹⁾

The finite element discretization of the system results in a diagonal mass matrix, a tridiagonal damping and stiffness matrices of the following forms:

$$M = mI \quad (15)$$

$$C = \alpha M + \beta K \quad (16)$$

$$K = k \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \quad (17)$$

where α and β are the damping coefficients of the Rayleigh damping. The analytical solutions can be resulted through following relationships:

$$\lambda_{2i-1, 2i} = -\xi_i \omega_i + j\omega_i + \sqrt{1 - \xi_i^2} \quad \text{for } i=1, \dots, n \quad (18)$$

$$\xi_i = \frac{1}{2} \left(\frac{\alpha}{\omega_i} + \beta \omega_i \right) \quad (19)$$

$$\omega_i = 2\sqrt{\frac{m}{k}} \sin \frac{2i-1}{2n+1} \frac{\pi}{2} \quad (20)$$

where ω_i and ξ_i are the undamped natural frequency and modal damping ratio, respectively.

A system with order 10 is used in analysis. The k and m are 1, and the coefficients, α and β , of the Rayleigh damping are 0.05 and 0.5, respectively. All the eigenvalues and their radius from the origin in the complex plane are as in Table I. In this example the checking process performed for open disks of three different radius.

First two cases are for checking in between eigenvalues. Their location is selected considering the relative distances between two adjacent eigenvalues except for a conjugate eigenvalues, and two eigenvalues with smaller distances from the next are selected. The third is for checking the number of all the eigenvalues of the system. The radius of the open disk is should be only a bit larger than the selected eigenvalue to ensure that the next eigenvalue is not within the open disk. The Sturm sequence check proposed by Bathe⁽¹⁾ has used 1.01 times the magnitude of the largest known eigenvalue for the eigenproblem without the damping matrix. However, according to Jung et al.⁽⁸⁾, the value used by Bathe is too large to ensure that the next largest eigenvalue is not within the open disk, and recommended times the magnitude of the largest known eigenvalue. In this example, therefore, the radius ρ of the open disk is chosen by 1.005 times the magnitude of the largest eigenvalue ($\rho = 1.005|\lambda|$). For each cases the calculated coefficient of the characteristic polynomial $\overline{a_i}$, diagonal element d_{ii} , and sign of d_i are as in Table I. Using the sign of d_i Table I, the number of eigenvalues for each cases are calculated as followings:

Case 1: $\rho = 1.005|\lambda_{16}| = 1.8109$

$$N_\lambda = 2n - S[1, d_1, d_2, \dots, d_{2n}] = 20 - 4 = 16$$

Case 2: $\rho = 1.005|\lambda_{18}| = 1.9207$

$$N_\lambda = 2n - S[1, d_1, d_2, \dots, d_{2n}] = 20 - 2 = 18$$

Case 4: $\rho = 1.005|\lambda_{20}| = 1.9875$

$$N_\lambda = 2n - S[1, d_1, d_2, \dots, d_{2n}] = 20 - 0 = 20$$

Referring to Table I, the number of eigenvalues which is inside open disks of radius $\rho = 1.005|\lambda_{16}| = 1.8109$, $\rho = 1.005|\lambda_{18}| = 1.9207$ and $\rho = 1.005|\lambda_{20}| = 1.9875$ are 16, 18 and 20 which are exactly agree with the calculated values.

As seen from this result, therefore, we verify that the proposed method can exactly check the number of eigenvalues inside some open disk of arbitrary radius.

4. CONCLUSIONS

A technique of calculating the number of eigenvalues inside an open disk of arbitrary radius was given. The technique is based on Chens algorithm and Gleyse's theorem and can be used to check the missed eigenvalues for the eigenproblem with damping matrix. By analyzing the numerical examples, it is verified that the proposed method can exactly check the number of eigenvalues for distinct or multiple eigenvalues for damped systems.

The technique by Jung et al. should find the variation of arguments of complex numbers along a predefined path. Therefore, a large number of checking points should be used to obtain accurate result. However, the proposed method can exactly find the number of eigenvalues by performing the factorization process only once. In result, much effort in finding the number of eigenvalue of larger structures with damping matrix can be eliminated by the proposed method.

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Table I. The calculated eigenvalues, characteristic coefficient \bar{a}_i and sign of d_i

i	Eigenvalues(λ)		Radius ($ \lambda $)	$\rho=1.005 \lambda_{16} = 1.8109$		$\rho=1.005 \lambda_{18} = 1.9207$		$\rho=1.005 \lambda_{20} = 1.9875$	
	Real	Imag.		\bar{a}_i	$sign(d_i)$	\bar{a}_i	$sign(d_i)$	\bar{a}_i	$sign(d_i)$
0	-	-	-	3.079e-04	+	2.055e-04	+	1.618e+00	+
1	-0.0306	-0.1463	0.1495	4.322e-03	+	3.059e-03	+	2.492e+00	+
2	-0.0306	0.1463	0.1495	8.065e-02	+	6.055e-02	+	5.104e+00	+
3	-0.0745	-0.4388	0.4450	6.252e-01	+	4.979e-01	+	4.343e+00	+
4	-0.0745	0.4388	0.4450	4.147e+00	+	3.503e+00	+	3.161e+00	+
5	-0.1585	-0.7133	0.7307	1.947e+00	+	1.745e+01	+	1.629e+01	+
6	-0.1585	0.7133	0.7307	7.601e+01	+	7.222e+01	+	6.979e+01	+
7	-0.2750	-0.9614	1.0000	2.373e+02	+	2.391e+02	+	2.391e+02	+
8	-0.2750	0.9614	1.0000	6.251e+02	+	6.681e+02	+	6.913e+02	+
9	-0.4137	-1.1763	1.2470	1.374e+03	+	1.558e+03	+	1.668e+03	+
10	-0.4137	1.1763	1.2470	2.581e+03	+	3.104e+03	+	3.439e+03	+
11	-0.5624	-1.3540	1.4661	4.111e+03	+	5.242e+03	+	6.011e+03	+
12	-0.5624	1.3540	1.4661	5.610e+03	+	7.588e+03	+	9.002e+03	+
13	-0.7077	-1.4932	1.6525	6.495e+03	-	9.318e+03	+	1.144e+04	+
14	-0.7077	1.4932	1.6525	6.395e+03	+	9.731e+03	+	1.236e+04	+
15	-0.8368	-1.5959	1.8019	5.261e+03	+	8.491e+03	+	1.116e+04	+
16	-0.8368	1.5959	1.8019	3.592e+03	+	6.149e+03	+	8.365e+03	+
17	-0.9381	-1.6651	1.9111	1.960e+03	-	3.558e+03	-	5.008e+03	+
18	-0.9381	1.6651	1.9111	8.325e+02	+	1.603e+03	+	2.335e+03	+
19	-1.0028	-1.7046	1.9777	2.446e+02	+	4.996e+02	+	7.529e+02	+
20	-1.0028	1.7046	1.9777	4.429e+01	+	9.595e+01	+	1.496e+02	+

* $sign(d_i)$ represents the sign of d_i , so $sign(d_i)$ is '+' when $d_i > 0$ and $sign(d_i)$ is '-' when $d_i < 0$.
 d_0 is defined as equal to 1.

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