감쇠시스템을 위한 개선된 Sturm 수열 성질

Modified Sturm Sequence Property for Damped Systems

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1. INTRODUCTION

To obtain the dynamic response of a structure, it is economic and efficient to superpose the results of a few lowest modes. But some important modes can be missed in the calculation process, which may lead to poor results in dynamic analysis. Hence, a checking technique for missed eigenvalues is required in finding the missed one. For the case of undamped system or proportionally damped system, it can be easily found by using the Sturm sequence property [1].

In the case of the non- proportionally damped systems, no counterpart of the Sturm sequence property for undamped systems has been developed yet [2]. Hence, when some important modes are missed for those systems, it may leads to poor results in dynamic analysis.

In this paper, Gleyse's theorem [3], which can count the number of zeros of a characteristic polynomial inside an open unit disk, is used to calculate the number of eigenvalues for eigenproblem with the damping matrix.

2. MODIFIED STURM SEQUENCE PROPERTY FOR DAMPED SYSTEMS

2.1. The equations of motion of damped systems

The equation of motion of damped systems can be written as:

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = 0, \tag{1}$$

where M, K and C are the $(n\times n)$ mass, stiffness and nonclassical damping matrices, respectively, and $\ddot{u}(t)$, $\dot{u}(t)$ and $u(t)$ are the $(n\times 1)$ acceleration, velocity and displacement vectors, respectively. To find the solution of the free vibration of the system, we consider the following quadratic eigenproblem:

$$\lambda^2 M\phi + \lambda C\phi + K\phi = 0, \tag{2}$$

in which λ and ϕ are the eigenvalue and eigenvector of the system. There are $2n$ eigenvalues for the system with n degrees of freedom and these occur either in real pairs or in complex conjugate pairs, depending upon whether they correspond to overdamped or undamped modes.

The common practice is to reformulate the quadratic system of equation to a linear one by doubling the order of the system [5] such as:

$$\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix} = \lambda \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix} \tag{3}$$

In general, M is nonsingular, that is, $det(M)\neq 0$, so the above equation can be changed to the form of a standard eigenproblem:

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$$A\psi = \lambda\psi \quad (4), \qquad A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \psi = \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix}. \quad (5)$$

The characteristic polynomial of Equation (4) can be represented as:

$$P(\lambda) = \det(A - \lambda I) = a_{2n}\lambda^{2n} + a_{2n-1}\lambda^{2n-1} + \cdots + a_1\lambda + a_0 = \sum_{i=0}^{2n} a_i\lambda^i = 0, \quad (6)$$

where λ is a complex value and $a_i (i = 0, 1, \cdots, 2n)$ are the real coefficients.

2.2. The coefficient of the characteristic polynomial

Chen [4] suggested a stable algorithm to obtain the coefficients of the characteristic polynomial of a real square matrix. According to his algorithm some given matrix A, can be transformed to its companion form $\overline{A}$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,2n-1} & a_{1,2n} \\ a_{21} & a_{22} & \cdots & a_{2,2n-1} & a_{2,2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2n-1,1} & a_{2n-1,2} & \cdots & a_{2n-1,2n-1} & a_{2n-1,2n} \\ a_{2n,1} & a_{2n,2} & \cdots & a_{2n,2n-1} & a2_{n,2n} \end{bmatrix}, (7) \qquad \overline{A} = \begin{bmatrix} -a_{2n-1} & -a_{2n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad (8)$$

by applying a sequence of Gauss-elimination like similarity transformations. When some numerical instability occurs during the transformations, modified algorithm by Chen [4] can be used. $P(\lambda) = det(A - \lambda I)$ can be obtained by observing the transformed matrix $\overline{A}$, and the characteristic polynomial is:

$$P(\lambda) = \det(A - \lambda I) = \det(\overline{A} - \lambda I) = \lambda^{2n} + a_{2n-1}\lambda^{2n-1} + \cdots + a_1\lambda + a_0 = 0, \ (a_{2n} = 1) \quad (9)$$

2.3. The number of eigenvalues in an open unit disk.

Gleyse [3] suggested a method of checking the number of eigenvalues of a real polynomial inside an open unit disk by a determinant representation.

Let $P(\lambda) = \sum_{h=0}^{2n} a_h \lambda_h = 0$ (a_h is a real number), then the number of eigenvalues inside an open unit disk can be determined as:

$$N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}] \quad (10)$$

where N_λ is the number of eigenvalues in an open unit disk, 2n is the degree of the polynomial, $S[k_0, k_1, k_2, \cdots, k_{2n}]$ is the number of sign changes in the sequence $k_i (i = 0, 1, \cdots, 2n)$ and $d_i (i = 1, 2, \cdots, 2n)$ is the determinants(minors) of the leading principal submatrices of order i in the Schur-Cohn matrix T:

$$T = \begin{bmatrix} \ddots & & \\ & t_{ij} & \\ & & \ddots \end{bmatrix}, \quad t_{ij} = \sum_{h=0}^{\min(i,j)} \left(a_{2n-i+h} a_{2n-j+h} - a_{i-h} a_{j-h} \right). \quad (11)$$

2.4. Modified Sturm sequence property

To apply Gleyse's theorem [3] for an open disks of arbitrary radius ρ, substitute $\lambda = \rho\overline{\lambda}$(ρ is an real number) to Equation (6), then:

$$P(\overline{\lambda}) = a_{2n}\rho^{2n}\overline{\lambda}^{2n} + a_{2n-1}\rho^{2n-1}\overline{\lambda}^{2n-1} + \cdots + a_1\rho\overline{\lambda} + a_0 = \overline{a}_{2n}\overline{\lambda}^{2n} + \overline{a}_{2n-1}\overline{\lambda}^{2n-1} + \cdots + \overline{a}_1\overline{\lambda} + \overline{a} = \sum_{i=0}^{2n} \overline{a}_i\overline{\lambda}^i = 0, \quad (12)$$

where $\overline{a_i} = a_i\rho^i (i = 0, 1, \cdots, 2n)$ are modified coefficients.

Using the modified coefficients $\overline{a_i}$ $(i=0,1,\cdots,2n)$ in Equation (12), His theorem can be extended to calculate the number of eigenvalues in the open disks of arbitrary radius ρ. The calculation of d_i $(i=1,2,\cdots,2n)$ can be easily performed by the LDL^T factorization of the Schur-Cohn matrix T. If $T = LDL^T$, then:

$$T_i = L_i D_i L_i^T \tag{13}$$

where T_i is the leading principal submatrices of order i in the T, L_i is the leading principal submatrices of order i in the L and D_i is the leading principal submatrices of order i in the D. Therefore each d_i $(i=1,2,\cdots,2n)$ can be easily obtained as:

$$d_i = \det(T_i) = \det(L_i D_i L_i^T) = \det(D_i) = d_{11} \times d_{22} \times \Lambda \times d_{ii} = \prod_{h=1}^{i} d_{hh} \tag{14}$$

Considering Equation (10), we only need to know the signs of each d_i because the unknown value of $S[1, d_1, d_2, \cdots, d_{2n}]$ depends only on sign changes of each $d_i(i=1,2,\cdots,2n)$ and from Equation (14) the signs of each d_i can be determined from the number of negative elements of each diagonal elements of D_i, which is very similar to Strum sequence property for undamped systems.

3. SINGLE SPRING-MASS-DAMPER SYSTEM[5] EXAMPLE

The finite element discretization of the system results in a diagonal mass matrix, a tridiagonal damping and stiffness matrices of the following forms:

$$M = mI \quad (15), \qquad C = \alpha M + \beta K \quad (16), \qquad K = k\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \text{O} & \text{O} & \\ & & \text{O} & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \quad (17)$$

where α and β are the damping coefficients of the Rayleigh damping. The analytical solutions can be resulted through following relationships:

$$\lambda_{2i-1,2i} = -\xi_i\omega_i + j\omega_i + \sqrt{1-\xi_i^2} \ \text{for} \ i=1,\cdots,n \ (18), \quad \xi_i = \frac{1}{2}\left(\frac{\alpha}{\omega_i} + \beta\omega_i\right) \ (19), \quad \omega_i = 2\sqrt{\frac{m}{k}} \sin\frac{2i-1}{2n+1}\frac{\pi}{2} \ (20)$$

where ω_i and ξ_i are the undamped natural frequency and modal damping ratio, respectively.

A system with order 10 is used in analysis. The k and m are 1, and the coefficients, α and β, of the Rayleigh damping are 0.05 and 0.5, respectively. The radius ρ of the open disk is chosen by 1.005 times the magnitude of the largest eigenvalue($\rho=1.005|\lambda|$). For each cases the calculated coefficient of the characteristic polynomial $\overline{a_i}$, diagonal element d_{ii}, and and sign of d_i are as in Table I. Using the sign of d_i in Table I, the number of eigenvalues for each cases are calculated as followings:

Case 1: $\rho=1.005|\lambda_{16}|$ = 1.8109, $N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}] = 20 - 4 = 16$

Case 2: $\rho=1.005|\lambda_{18}|$ = 1.9207, $N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}] = 20 - 2 = 18$

Case 4: $\rho=1.005|\lambda_{20}|$ = 1.9875, $N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}] = 20 - 0 = 20$

Referring to Table I, the above results are exactly agree with the calculated values.

4. CONCLUSIONS

A technique of calculating the number of eigenvalues inside an open disk of arbitrary radius was given. The technique is based on Chens algorithm and Gleyse's theorem and can be used to check the missed eiegnvalues for the eigenproblem with damping matrix. By analyzing the numerical examples, it is verified that the proposed method can exactly check the number of eigenvalues for distinct or multiple eigenvalues for damped systems.

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Table I. The calculated eigenvalues, characteristic coefficient $\overline{a_i}$ and sign of d_i

i	Eigenvalues(λ)		Radius	$\rho=1.005\lvert\lambda_{16}\rvert$ = 1.8109		$\rho=1.005\lvert\lambda_{18}\rvert$ = 1.9207		$\rho=1.005\lvert\lambda_{20}\rvert$ = 1.9875	
	Real	Imag.	($\lvert\lambda\rvert$)	$\overline{a_i}$	$sign(d_i)$	$\overline{a_i}$	$sign(d_i)$	$\overline{a_i}$	$sign(d_i)$
0	-	-	-	3.079e-04	+	2.055e-04	+	1.618e+00	+
1	-0.0306	-0.1463	0.1495	4.322e-03	+	3.059e-03	+	2.492e+00	+
2	-0.0306	0.1463	0.1495	8.065e-02	+	6.055e-02	+	5.104e+00	+
3	-0.0745	-0.4388	0.4450	6.252e-01	+	4.979e-01	+	4.343e+00	+
4	-0.0745	0.4388	0.4450	4.147e+00	+	3.503e+00	+	3.161e+00	+
5	-0.1585	-0.7133	0.7307	1.947e+00	+	1.745e+01	+	1.629e+01	+
6	-0.1585	0.7133	0.7307	7.601e+01	+	7.222e+01	+	6.979e+01	+
7	-0.2750	-0.9614	1.0000	2.373e+02	+	2.391e+02	+	2.391e+02	+
8	-0.2750	0.9614	1.0000	6.251e+02	+	6.681e+02	+	6.913e+02	+
9	-0.4137	-1.1763	1.2470	1.374e+03	+	1.558e+03	+	1.668e+03	+
10	-0.4137	1.1763	1.2470	2.581e+03	+	3.104e+03	+	3.439e+03	+
11	-0.5624	-1.3540	1.4661	4.111e+03	+	5.242e+03	+	6.011e+03	+
12	-0.5624	1.3540	1.4661	5.610e+03	+	7.588e+03	+	9.002e+03	+
13	-0.7077	-1.4932	1.6525	6.495e+03	-	9.318e+03	+	1.144e+04	+
14	-0.7077	1.4932	1.6525	6.395e+03	+	9.731e+03	+	1.236e+04	+
15	-0.8368	-1.5959	1.8019	5.261e+03	+	8.491e+03	+	1.116e+04	+
16	-0.8368	1.5959	1.8019	3.592e+03	+	6.149e+03	+	8.365e+03	+
17	-0.9381	-1.6651	1.9111	1.960e+03	-	3.558e+03	-	5.008e+03	+
18	-0.9381	1.6651	1.9111	8.325e+02	+	1.603e+03	+	2.335e+03	+
19	-1.0028	-1.7046	1.9777	2.446e+02	+	4.996e+02	+	7.529e+02	+
20	-1.0028	-1.7046	1.9777	4.429e+01	+	9.595e+01	+	1.496e+02	+

* $sign(d_i)$ represents the sign of d_i , so $sign(d_i)$ is '+' when $d_i > 0$ and $sign(d_i)$ is '-' when $d_i < 0$.
d_0 is defined as equal to 1.