StepLength를이용한비비례감쇠시스템의고유치해석

Application of Step Length Technique to An Eigensolution Method for Non-proportionally Damped Systems

Thanh X. Nguyen* · 김병완** · 정형조*** · 이인원****

Thanh X. Nguyen · Kim, Byoung Wan · Jung, Hyung Jo · Lee, In Won

1. INTRODUCTION

The eigenvalue problem should be solved a priori to avoid a resonance or to obtain the dynamic characteristics such as natural frequencies and mode shapes if the mode superposition method is used in the dynamic analysis of structures. In most recently employed analyses, the proportional damping is assumed unrealistically to be such that the free vibration modes of the damped system are identical to those of the undamped one. However, in most real systems, damping is non-proportional. Even when proportional damping is assumed for each sub-system in the analysis of soil-structure systems, composite structures, etc., the resulting damping for the complete structure will still be nonproportional.

Researchers have proposed many eigensolution methods (Tisseur and Meerbergen 2001). The subspace iteration method (Bathe and Wilson 1972), the Lanczos method (Lanczos 1950) and the methods based on Newton scheme (Peters and Wilkinson 1979; Lee *et al.* 1998; Fokkema *et al.* 1998) are widely used recently.

The subspace iteration method combines the inverse iteration method, simultaneous iteration method and Rayleigh-Ritz analysis. It is a more efficient alternative algorithm than the inverse iteration procedure. The method employs *n*order submatrices of the augmented linear eigenproblem in the iteration process by taking the block-partitioned nature of the matrices of the linearized problem. All required modes are solved simultaneously thus the round-off errors can be minimized. However, it requires a great deal of complex arithmetic.

Lanczos method (Lanczos 1950) was first proposed for undamped systems and then developed fordamped systems. The two sided-Lanczos algorithm requires the generation of two sets of Lanczos vectors, left and right, and the symmetric Lanczos algorithm uses a set of Lanczos vectors to reduce a large eigenvalue problem in a much smaller one. Although only real arithmetic is used in the solution process, in contrast to the case of real symmetric eigenproblems, there will be a possibility of a serious breakdown and the accuracy of the solutions obtained is low.

Having a detailed reference list, Lee *et al*. (1998) reviewed almost of the work above and proposed anefficient solution method in order to improve the numerical stability and increase the convergence by applying the modified Newton-Raphson technique and the orthonormal condition of the eigenvectors. To further improve the convergence of the method, the accelerated scheme adopting the step length that can be evaluated by using the least squares technique is proposed in this study. In the following section, the basic concept of the proposed method is presented. In section 3, the efficiency of the proposed method is shown via a numerical example.

* Ph.D. Candidate, Dept. of Civil and Environmental Engrg., KAIST · E-mail: txn99@hotmail.com - Speaker
** Dr., Dept. of Civil and Environmental Engrg., KAIST · E-mail: kimbw@kaist.ac.kr
*** Member, Ass. Prof., Dept. of Civil and Environmental Engrg., · Sejong Univ. · E-mail: hjung@sejong.ac.kr
**** Member, Prof., Dept. of Civil and Environmental Engrg., KAIST · E-mail: iwlee@kaist.ac.kr

2. METHOD OF ANALYSIS

2.1 Problem Definition

The equation for free vibration of a linear time-invariant system of order *n* is written as

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \tag{1}$$

where **M**, **C** and **K** are $(n \times n)$ mass, damping and stiffness matrices of the system, respectively and **u** is the $(n \times 1)$ global displacement vector. The damping is said to be proportional if

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C} \tag{2}$$

In this case, the eigenpairs of the system (1) could be found in a straightforward and efficient manner. However, generally, the damping matrix does not satisfy Eq. (2), that is, it is non-proportional. The eigenanalysis for such systems is traditionally performed in the space extended to *2n*-dimensional such as

$$\begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\phi} \\ \lambda\boldsymbol{\phi} \end{Bmatrix} = \lambda \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\phi} \\ \lambda\boldsymbol{\phi} \end{Bmatrix} \tag{3}$$

where and are eigenvalue and associated eigenvector of the system, respectively. Eq. (3) may be written as

$$\mathbf{A}\boldsymbol{\psi} = \lambda\mathbf{B}\boldsymbol{\psi} \tag{4}$$

with

$$\mathbf{A} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \text{ and } \quad \boldsymbol{\psi} = \begin{Bmatrix} \boldsymbol{\phi} \\ \lambda\boldsymbol{\phi} \end{Bmatrix} \tag{5}$$

Since both matrices **A** and **B** are not positive definite although they are symmetric, in general, the eigenpairs are complex values.

2.2 Modified Newton-Raphson method (Lee *et al.* 1998)

Suppose that initial approximate solutions $\lambda^{(0)}$ and $\boldsymbol{\psi}^{(0)}$ of the eigenvalue and the associated eigenvector of Eq. (5) are known. Denote the approximate eigenvalue after the *k*th iteration by $\lambda^{(k)}$ and its associated eigenvector by $\boldsymbol{\psi}^{(k)}$, and define the residual vector as

$$\mathbf{r}^{(k)} = \mathbf{A}\boldsymbol{\psi}^{(k)} - \lambda^{(k)}\mathbf{B}\boldsymbol{\psi}^{(k)} \tag{6}$$

The approximate eigenvector $\boldsymbol{\psi}^{(k)}$ is then orthonormalized with respect to matrix **B**, such as

$$\left(\boldsymbol{\psi}^{(k)}\right)^T \mathbf{B}\boldsymbol{\psi}^{(k)} = 1 \tag{7}$$

where *T* stands for the transpose operator and thus $\left(\boldsymbol{\psi}^{(k)}\right)^T$ means the transpose of $\boldsymbol{\psi}^{(k)}$.

Let the increment of the approximate eigenvalue from step *k* to step (*k* + 1) be $\Delta\lambda^{(k)}$, and the increment vector of the approximate eigenvector from step *k* to step (*k* + 1) be $\Delta\boldsymbol{\psi}^{(k)}$.

$$\lambda^{(k+1)} = \lambda^{(k)} + \Delta\lambda^{(k)} \tag{8}$$

$$\boldsymbol{\psi}^{(k+1)} = \boldsymbol{\psi}^{(k)} + \Delta\boldsymbol{\psi}^{(k)} \tag{9}$$

After the (*k* + 1)th iteration, the residual vector is

$$\mathbf{r}^{(k+1)} = \mathbf{A}\boldsymbol{\psi}^{(k+1)} - \lambda^{(k+1)}\mathbf{B}\boldsymbol{\psi}^{(k+1)} \tag{10}$$

where $\boldsymbol{\psi}^{(k+1)}$ also satisfies the orthonormal condition with respect to matrix **B** as follows

$$\left(\boldsymbol{\psi}^{(k+1)}\right)^T \mathbf{B}\boldsymbol{\psi}^{(k+1)} = 1 \tag{11}$$

Substitute Eqs. (8), (9) into Eq. (10), we have

$$\mathbf{r}^{(k+1)} = \left[\mathbf{A} - \left(\lambda^{(k)} + \Delta\lambda^{(k)}\right)\mathbf{B}\right]\left(\boldsymbol{\psi}^{(k)} + \Delta\boldsymbol{\psi}^{(k)}\right) \tag{12}$$

To get the solution converged to the eigenvalue and its associated eigenvector, we expect the residual vector to be a zero vector, such as

$$\left[\mathbf{A} - \left(\lambda^{(k)} + \Delta\lambda^{(k)}\right)\mathbf{B}\right]\left(\boldsymbol{\psi}^{(k)} + \Delta\boldsymbol{\psi}^{(k)}\right) = \mathbf{0} \tag{13}$$

Introducing Eqs. (6) to (9) and neglecting the high order terms, namely, $\Delta\lambda^{(k)}\mathbf{B}\Delta\boldsymbol{\psi}^{(k)}$ and $\left(\Delta\boldsymbol{\psi}^{(k)}\right)^T\mathbf{B}\Delta\boldsymbol{\psi}^{(k)}$, Eqs. (11) and (13) can be rewritten as

$$\left(\mathbf{A} - \lambda^{(k)}\mathbf{B}\right)\Delta\boldsymbol{\psi}^{(k)} - \Delta\lambda^{(k)}\mathbf{B}\boldsymbol{\psi}^{(k)} = -\mathbf{r}^{(k)} \tag{14}$$

$$\left(\boldsymbol{\psi}^{(k)}\right)^T\mathbf{B}\Delta\boldsymbol{\psi}^{(k)} = 0 \tag{15}$$

Eqs. (14) and (15) with two unknowns $\Delta\lambda^{(k)}$ and $\Delta\boldsymbol{\psi}^{(k)}$ form the simultaneous equations system which can be written in matrix form as

$$\begin{bmatrix} \mathbf{A} - \lambda^{(k)}\mathbf{B} & -\mathbf{B}\boldsymbol{\psi}^{(k)} \\ -\left(\boldsymbol{\psi}^{(k)}\right)^T\mathbf{B} & 0 \end{bmatrix}\begin{Bmatrix} \Delta\boldsymbol{\psi}^{(k)} \\ \Delta\lambda^{(k)} \end{Bmatrix} = -\begin{Bmatrix} \mathbf{r}^{(k)} \\ 0 \end{Bmatrix} \tag{16}$$

The symmetric coefficient matrix of this linear algebraic equation is of order ($2n$ + 1). If all eigenvalues are distinct $\lambda_i \neq \lambda_j \ (i \neq j)$, then the coefficient matrix is nonsingular. The method using the Newton-Raphson technique, despite its rapid convergence, is not efficient because a new coefficient matrix has to be reformed and refactorized in each iteration step (Lee *et al.* 1998). This time-consuming procedure could be avoided by applying the modified Newton-Raphson technique as follows

$$\begin{bmatrix} \mathbf{A} - \lambda^{(0)}\mathbf{B} & -\mathbf{B}\boldsymbol{\psi}^{(k)} \\ -\left(\boldsymbol{\psi}^{(k)}\right)^T\mathbf{B} & 0 \end{bmatrix}\begin{Bmatrix} \Delta\boldsymbol{\psi}^{(k)} \\ \Delta\lambda^{(k)} \end{Bmatrix} = -\begin{Bmatrix} \mathbf{r}^{(k)} \\ 0 \end{Bmatrix} \tag{17}$$

The symmetric coefficient matrix of (17) is also nonsingular. Once the submatrix $\mathbf{A} - \lambda^{(0)}\mathbf{B}$ is decomposed into $\mathbf{LDL}^T$ (**L**: lower triangular matrix, **D**: diagonal matrix), a small number of operations are required to solve (17), since only the $\mathbf{B}\boldsymbol{\psi}^{(k)}$ in the coefficient matrix is changed in each iteration. However, due to negligence of the small nonlinear term $\left(\lambda^{(k+1)} - \lambda^{(0)}\right)\mathbf{B}\Delta\boldsymbol{\psi}^{(k)}$, the convergenceis lower. Therefore, the improvement of the convergence of the method is needed to apply to large-scale system.

2.3 Proposed method

To further improve the convergence of the method, the step length and a selective scheme are introduced in this study. Since the convergence rate of the eigenvector is lower than that of the eigenvalue, the accelerated scheme is applied only to the eigenvector as follows

$$\boldsymbol{\psi}^{(k+1)} = \boldsymbol{\psi}^{(k)} + \alpha^{(k)}\Delta\boldsymbol{\psi}^{(k)} \tag{18}$$

Because Eq. (18) is introduced instead of Eq. (9), the residual vector might not be a zero vector as in Eq. (13). To minimize its norm, the least square technique is used as follows

$$\frac{\partial}{\partial\alpha^{(k)}}\left(\left(\bar{\mathbf{r}}^{(k+1)}\right)^H\bar{\mathbf{r}}^{(k+1)}\right) = 0 \tag{19}$$

where

$$\bar{\mathbf{r}}^{(k+1)} = \left(\mathbf{A} - \lambda^{(k+1)}\mathbf{B}\right)\left(\boldsymbol{\psi}^{(k)} + \alpha^{(k)}\Delta\boldsymbol{\psi}^{(k)}\right) \tag{20}$$

and H stands for the Hermitian transpose operator and thus $\left(\bar{\mathbf{r}}^{(k+1)}\right)^H$ is the complex conjugate of the transpose of $\bar{\mathbf{r}}^{(k+1)}$.

Solving Eq. (19) for $\alpha^{(k)}$ yields

$$\alpha^{(k)} = -\frac{\left(\boldsymbol{\psi}^{(k)}\right)^H \left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)^H \left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)\Delta\boldsymbol{\psi}^{(k)} + \left(\Delta\boldsymbol{\psi}^{(k)}\right)^H \left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)^H \left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)\boldsymbol{\psi}^{(k)}}{2\left(\Delta\boldsymbol{\psi}^{(k)}\right)^H \left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)^H \left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)\Delta\boldsymbol{\psi}^{(k)}} \tag{21}$$

Note that $\lambda^{(k+1)}$ and $\Delta\boldsymbol{\psi}^{(k)}$ have been obtained in Eqs. (8) and (17), respectively. If $\alpha^{(k)} \approx 1.0$, it cannot accelerate convergence. In that case, the proposed method is not more efficient than the conventional method since $\alpha \approx 1.0$ does not contribute much to the whole procedure. Therefore, selective application of α is required. According to our experience, it would be better to not to apply α in the first step. The error norm defined in Eq. (23) after the first step is used to determine whether to apply α from the second step on. Thus, we introduce a checking number γ. If the error norm after the first step is greater than γ, step length is then applied and vice versa. Eqs. (8), (17), (18) and (21) form the basic algorithm of the proposed method.

Initial values of the proposed method can be obtained from the intermediate results of the iteration methods or from the results of the approximate methods. In this paper, the starting values are taken as the results of the symmetric Lanczos method. The algorithm of the proposed method is shown in Table 1.

Table 1. Algorithm of the proposed method

1. Calculate initial eigenvalue $\lambda^{(0)}$ and eigenvectors $\boldsymbol{\psi}^{(0)}$.
2. For each eigenvalue and the associated eigenvector, apply the following procedure:
   (a) Perform the 1st step by the conventional method.
   Calculate the error norm. If error norm > γ (checking number), '*apply*' = 'Yes'
   (b) For k = 1, 2, 3, ...
   (c) Define $\begin{bmatrix} \mathbf{A}-\lambda^{(0)}\mathbf{B} & -\mathbf{B}\boldsymbol{\psi}^{(k)} \\ -\left(\boldsymbol{\psi}^{(k)}\right)^T\mathbf{B} & 0 \end{bmatrix}$
   Define $-\begin{Bmatrix} \mathbf{r}^{(k)} \\ 0 \end{Bmatrix}$ where $\mathbf{r}^{(k)} = \mathbf{A}\boldsymbol{\psi}^{(k)} - \lambda^{(k)}\mathbf{B}\boldsymbol{\psi}^{(k)}$
   Solve $\begin{bmatrix} \mathbf{A}-\lambda^{(0)}\mathbf{B} & -\mathbf{B}\boldsymbol{\psi}^{(k)} \\ -\left(\boldsymbol{\psi}^{(k)}\right)^T\mathbf{B} & 0 \end{bmatrix}\begin{Bmatrix} \Delta\boldsymbol{\psi}^{(k)} \\ \Delta\lambda^{(k)} \end{Bmatrix} = -\begin{Bmatrix} \mathbf{r}^{(k)} \\ 0 \end{Bmatrix}$ for $\begin{Bmatrix} \Delta\boldsymbol{\psi}^{(k)} \\ \Delta\lambda^{(k)} \end{Bmatrix}$
   (d) Compute $\lambda^{(k+1)} = \lambda^{(k)} + \Delta\lambda^{(k)}$
   If '*apply*'= 'Yes' then

$$\alpha^{(k)} = -\frac{\left(\boldsymbol{\psi}^{(k)}\right)^H \left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)^H \left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)\Delta\boldsymbol{\psi}^{(k)} + \left(\Delta\boldsymbol{\psi}^{(k)}\right)^H \left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)^H \left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)\boldsymbol{\psi}^{(k)}}{2\left(\Delta\boldsymbol{\psi}^{(k)}\right)^H \left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)^H \left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)\Delta\boldsymbol{\psi}^{(k)}}$$

   otherwise $\alpha^{(k)} = 1$ end
   $\boldsymbol{\psi}^{(k+1)} = \boldsymbol{\psi}^{(k)} + \alpha^{(k)}\Delta\boldsymbol{\psi}^{(k)}$
   Normalize $\boldsymbol{\psi}^{(k+1)}$ with respect to matrix $\mathbf{B}$
   $\mathbf{r}^{(k+1)} = \mathbf{A}\boldsymbol{\psi}^{(k+1)} - \lambda^{(k+1)}\mathbf{B}\boldsymbol{\psi}^{(k+1)}$

$$error\ norm = \frac{\left\|\mathbf{r}^{(k+1)}\right\|_2}{\left\|\mathbf{A}\boldsymbol{\psi}^{(k+1)}\right\|_2}$$

   (e) Update $\mathbf{A}-\lambda^{(0)}\mathbf{B}$ in (c) to $\mathbf{A}-\lambda^{(k)}\mathbf{B}$ if necessary to improve convergence
   (f) If $error\ norm \leq$ predetermined error limit, terminate the loop.

3. NUMERICAL EXAMPLE

A cantilever beam with multi-lumped translational viscous-dampers attached at nodes as shown in Fig. 1 is analyzed using two methods: the method proposed by Lee *et al.*(1998) and the proposed method. The structure has 100 elements and 200 degrees of freedom. The order of the associated (**A**, **B**) is 400. The inertia of the cross section I = 2.25E-8 m4. The cross section area A = 3.0E-4 m2. The density of the material = 8000 kg/m3. The Young modulus E = 2.0E+11 N/m2. The mass matrix **M** is a consistent one. The damping matrix **C** consists of the Rayleighdamping and the damping contributed from concentrated dampers. The Rayleigh damping is obtained by

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \tag{22}$$

where the Rayleigh coefficients α= 0.002 and β= 2.04E-7. The concentrated damping of the value 0.10 at each node is then added.

The error norm is computed by

$$error\ norm = \frac{\left\| \mathbf{A}\boldsymbol{\psi}^{(k)} - \lambda^{(k)}\mathbf{B}\boldsymbol{\psi}^{(k)} \right\|_2}{\left\| \mathbf{A}\boldsymbol{\psi}^{(k)} \right\|_2} \tag{23}$$

The error norm is compared with the predetermined error limit 1E-6. In this example, the checking number γ is assigned as $\gamma = 0.2$.

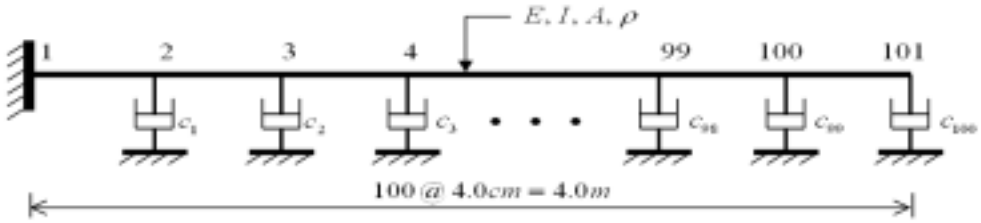

Fig. 1. Cantilever beam with lumped dampers

The total solution time to have 20 eigenpairs with the error norm of 1E-6 by using the proposed method and the method by Lee *et al.*(1998) is compared as follows. Letting the solution time using the proposed method be 1, then the method proposed by Lee *et al.*(1998) takes 1.08 times.

For each solution method, the convergence of the 14th and the 17th eigenpairs is depicted in Fig. 2. As shown in the figure, the convergence of the proposed method is better than that of Lee *et al.*(1998).

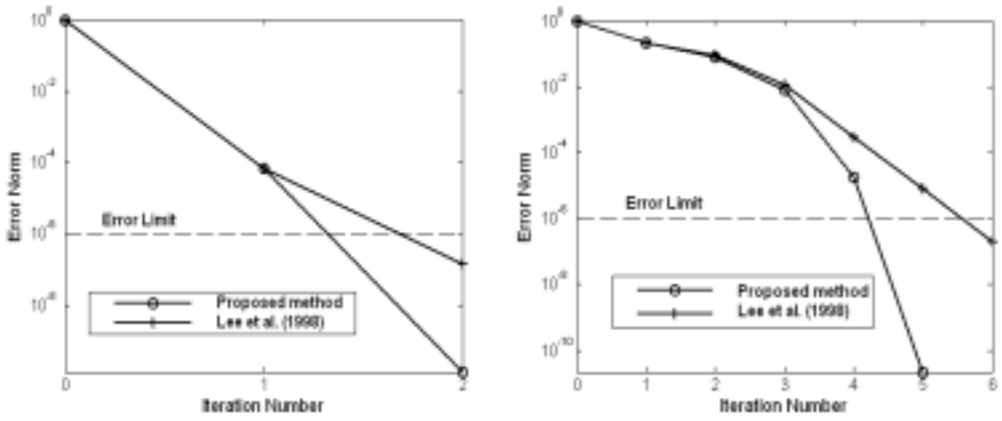

Fig. 2. Convergence of the 14th eigenpair (left) and the 17th eigenpair (right)

4. CONCLUSIONS

An efficient method for solving eigenproblems of the non-proportionally damped structures by applying the step length and the orthonormal side condition is presented. Characteristics of the proposed method illustrated by the numerical results are identified as

(1) The convergence rate of the proposed method is improved by introducing the step length.

(2) The algorithm of the proposed method is simple.

(3) The efficiency of the method depends on the checking number γ. Further study on the value of γ is being conducted.

ACKOWLEDGEMENT

The authors gratefully acknowledge the support of this research by the National Research Laboratory of Aseismic Control of Structures in Korea, the KOSEF and the Hyundai Engineering and Construction.

REFERENCES

1. Bathe, K. J., and Wilson, E. L. (1972). "Large eigenvalue problems in dynamic analysis." *J. Engrg. Mech.*, ASCE, 98, 1471-1485.

2. Fokkema, D. R., Sleijpen, G. L. G., Van der Vorst, H. A. (1998). "Accelerated inexact Newton schemes for large systems of nonlinear equations." *SIAM J. Sci. Comput.*, 19(2), 657-674.

3. Lanczos, C. (1950). "An iteration method for the solution of the eigenvalue problem of linear differential and integral operators." *J. Res. Nat. Bureau of Standards*, 45, 255-282.

4. Lee, I. W., Kim, M. C., and Robinson, A. R. (1998). "Efficient solution method of eigenproblems for damped structural systems using the modified Newton-Raphson technique." *J. Engrg. Mech.*, ASCE, 124(5), 576-580.

5. Peters, G., Wilkinson, J. H. (1979). "Inverse iteration, ill-conditioned equations and Newton's method." *SIAM Review*, 21(3), 339-360.

6. Tisseur, F., and Meerbergen, K. (2001). "The quadratic eigenvalue problem." *SIAM Review*, 43(2), 235-286.