구조물의 고유치 해석을 위한 개선된 Lanczos 방법

Improved Lanczos Method for the Eigenvalue Analysis of Structures

김병완* 김운학** 이인원***
Kim, Byoung-Wan Kim, Woon-Hak Lee, In-Won

ABSTRACT
This paper investigates the applicability of the modified Lanczos method using the power technique, which was developed in the field of quantum physics, to the eigenproblem in the field of engineering mechanics by introducing matrix-powered Lanczos recursion and numerically evaluating the suitable power value. The matrix-powered Lanczos method has better convergence and less operation count than the conventional Lanczos method. By analyzing four numerical examples, the effectiveness of the matrix-powered Lanczos method is verified and the appropriate matrix power is also recommended.

1. Introduction

Lanczos method\(^{(1)}\), has been known to be very efficient for the eigensolution method of structures. To improve the Lanczos method many researchers have studied a variety of procedures. Erricson and Ruhe\(^{(5)}\) have used shifting techniques to accelerate the Lanczos algorithm. Smith et al.\(^{(13)}\) have accelerated the Lanczos method through an implicitly restarted technique. Gambolati and Putti\(^{(6)}\) employed the preconditioned conjugate gradient scheme in the Lanczos method.
In the fields of quantum physics, Grosso et al.\(^{(7)}\) modified the Lanczos algorithm with powered operator to obtain the eigenstate of quantum systems. Similar power technique is found in accelerated subspace iteration method for structural dynamics.\(^{(10),(12),(14)}\) While, the modified Lanczos method using the power technique is not applied to structural dynamics yet. This paper applies the modified Lanczos method using the power technique to the eigenproblem of structural dynamics. In structural eigenproblem, the power technique can be applied to the matrix \(K^{-1}M\). The matrix \(K^{-1}M\) is called dynamic matrix.\(^{(4)}\) The modified Lanczos method using the power of dynamic matrix can accelerate the convergence of the conventional Lanczos method. Four numerical examples are presented to verify the effectiveness of the matrix–powered Lanczos method. The suitable power of the dynamic matrix in the method is also presented.

2. Matrix–powered Lanczos method

In the fields of quantum physics, Grosso et al.\(^{(7)}\) modified Lanczos recursion by introducing the second power of operator to accelerate the convergence as follows:

\[
b_{n+1}f_{n+1} = (H-E_i)^2f_n - a_nf_n - b_nf_{n-1}
\]  

(1)

where \(H\) is a given operator, \(f\) is basis functions, \(a\) and \(b\) are coefficients and \(n\) is Lanczos step number. \(E_i\) is trial energy which corresponds to shift in structural dynamics. The concept of power technique in (1) can be applied to the eigenproblem in structural dynamics. The eigenproblem of structure frequently encountered in structural dynamics can be expressed as

\[
K\phi_i = \lambda_i M\phi_i \quad (i = 1, 2, 3, \ldots, n)
\]  

(2)

where \(M\) and \(K\) are symmetric mass and stiffness matrices of order \(n\), respectively. \(\lambda_i\) and \(\phi_i\) are the \(i\)th eigenvalue and associated eigenvector of the system. To get the solution of (2), Lanczos schemed Ritz bases vectors through Gram–Schmidt orthogonalization of Krylov sequence as follows:\(^{(8)}\)

\[
x_{i+1} = (K^{-1}M)^jx_0 - \sum_{j=1}^{i} u_j x_j
\]  

(3)

where \(x_0\) is a starting vector, \(x_j\) is \(j\)th Lanczos vector, \(u_j\) is the component of \(v_i\) along \(x_\mu\), \(K_\mu = K - \mu M\) and \(\mu\) is shift. The concept of power technique can be applied to the dynamic matrix in (3), then following modified Gram–Schmidt orthogonalization can be introduced.

\[
x_{i+1} = ((K^{-1}M)^\delta)^jx_0 - \sum_{j=1}^{i} u_j x_j
\]  

(4)

Where \(\delta\) is positive integer. (4) means that an approximated eigenvector, whose number of
iteration is $\delta i$, is contained in $(i+1)$ Lanczos vectors. Whereas, in (3), $(i+1)$ Lanczos vectors contain an approximated eigenvector whose number of iterations is $i$. Therefore, (4) gives a better solution than (3). From (4), modified Lanczos recursion can be derived as

$$\tilde{x}_i = (K^{-1}_\mu M)^{\delta} x_i - \alpha_i x_i - \beta_i x_{i-1}$$

(5)

where $\alpha_i$ and $\beta_i$ are scalar coefficients obtained by

$$\alpha_i = x_i^T M (K^{-1}_\mu M)^{\delta} x_i, \quad \beta_i = (\tilde{x}_i^T M \tilde{x}_i)^{\frac{1}{2}}$$

(6)

then the next Lanczos vector is

$$x_{i+1} = \tilde{x}_i / \beta_i$$

(7)

With a set of Lanczos vectors, $X = [x_1 \ x_2 \ \ldots \ x_q]$, we can obtain the tridiagonalized standard eigenproblem of reduced order $q << n$

$$T\tilde{\phi}_i = (1/(\lambda_i - \mu)^{\delta})\tilde{\phi}_i \quad (i = 1, 2, 3, \ldots, q)$$

(8)

where

$$T = X^T M (K^{-1}_\mu M)^{\delta} X = \begin{bmatrix}
\alpha_1 & \beta_1 \\
\beta_1 & \alpha_2 & \beta_2 \\
& \ddots & \ddots & \ddots \\
& & \alpha_{q-1} & \beta_{q-1} \\
& & \beta_{q-1} & \alpha_q
\end{bmatrix}$$

(9)

The Lanczos algorithm is subjected to loss of orthogonality of Lanczos vectors due to round-off errors. In this paper, full reorthogonalization process is used to retain the orthogonality of the Lanczos vectors. The number of total operations for the matrix-powered Lanczos algorithm is

$$N_{\text{total}} = (1/2)nm^2 + (q^2 + 4q\delta + 5q + 3/2)nm + ((3/2)q^2 + q\delta + (17/2)q)n + 10q^2 + q + \sum_{j=2}^{q} js$$

(10)

where $n$ is system order, $m$ half-bandwidth, $q$ the number of calculated Lanczos vectors and $s$, the number of iterations of $j$th step in QR iteration for the eigenvalues of tridiagonal system.

### 3. Numerical examples

A simple spring–mass system with 100 DOFs, a plane framed structure, a three-dimensional frame structure and a three-dimensional building frame are analyzed to verify the effectiveness of the matrix–powered Lanczos method. With the predetermined error norm of $10^{-6}$, the number of operations for calculating desired eigenpairs is compared. To examine the
suitable power of dynamic matrix, numerical examples are analyzed with varying power of dynamic matrix. System matrices of a simple spring–mass system are

\[
M = \mathbf{I}, \quad K = \begin{bmatrix}
2 & -1 \\
-1 & 2 & -1 \\
& & \ddots \\
& & 2 & -1 \\
& & & -1 & 1
\end{bmatrix}
\]  

(11)

The geometric configurations and the material properties of a plane framed structure, a three-dimensional frame structure and a three-dimensional building frame are shown in Figs. 1 ~ 3.

![Fig. 1. Plane framed structure (DOFs: 330)](image1)

![Fig. 2. Three-dimensional frame structure (DOFs: 468)](image2)
Fig. 3. Three-dimensional building frame (DOFs: 1008)

Some results are shown in Table 1 and Fig. 4. The 1st power ($\delta = 1$) corresponds to the conventional Lanczos method. Table 1 and Fig. 4 show that the convergence of the matrix-powered Lanczos method is better than that of the conventional Lanczos method. However, in some cases, high matrix power causes failure in convergence due to the numerical instability.

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*: Failure in convergence due to numerical instability
Fig. 4. Number of operations for calculating desired eigenpairs

4. Conclusions

This paper investigates the applicability of the matrix-powered Lanczos method to the eigenproblem of structures. The characteristics of the matrix-powered Lanczos method by the numerical results from examples are summarized as follows:

(1) Since the power of the dynamic matrix can reduce the required number of Lanczos vectors, the matrix-powered Lanczos method has not only the better convergence but also the less operation count than the conventional Lanczos method.

(2) The suitable power of the dynamic matrix that gives numerically stable solution in the matrix-powered Lanczos method is the second power.
Acknowledgements

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References