가속화된 Lanczos 알고리즘을 이용한 구조물의 고유치 해법

Eigensolution Method for Structures Using Accelerated Lanczos Algorithm

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국문요약

본 논문에서는 양자물리학 분야에서 Lanczos 방법의 수렴을 가속화하기 위해 개발된 바 있는 행렬의 거듭제곱 기법을 동역학 분야의 Lanczos 순환식에 도입함으로써 구조물의 고유치 해석의 효율성을 향상시켰다. 행렬의 거듭제곱 기법을 도입한 Lanczos 방법이 기존의 방법보다 수렴성이 더욱 우수하다. 수치예제를 통해 행렬의 거듭제곱 기법을 도입한 Lanczos 방법의 효율성을 검증하였으며 제안방법을 통한 고유치 해석에 있어서 가장 적합한 거듭제곱값을 제시하였다.

1. Introduction

Lanczos method[11] has been known to be very efficient for the eigensolution method of structures. To improve the Lanczos method many researchers have studied a variety of procedures. Erricson and Ruhe[5] have used shifting techniques to accelerate the Lanczos algorithm. Smith et al.[13] have accelerated the Lanczos method through an implicitly restarted technique. Gambolati and Putti[6] employed the preconditioned conjugate gradient scheme in the Lanczos method.

In the fields of quantum physics, Grosso et al.[7] modified the Lanczos algorithm with powered operator to obtain the eigenstate of quantum systems. Similar power technique is found in accelerated subspace iteration method for structural dynamics.[10],[12],[14] While, the modified Lanczos method using the power technique is not applied to structural dynamics yet. This paper applies the modified Lanczos method using the power technique to the eigenproblem of structural dynamics. In structural eigenproblem, the power technique can be applied to the matrix $\mathbf{K}^{-1}\mathbf{M}$. The matrix $\mathbf{K}^{-1}\mathbf{M}$ is called dynamic matrix.[4]

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The modified Lanczos method using the power of dynamic matrix can accelerate the convergence of the conventional Lanczos method. Four numerical examples are presented to verify the effectiveness of the matrix-powered Lanczos method. The suitable power of the dynamic matrix in the method is also presented.

2. Matrix-powered Lanczos method

In the fields of quantum physics, Grosso et al.[7] modified Lanczos recursion by introducing the second power of operator to accelerate the convergence as follows:

$$b_{n+1}\mathbf{f}_{n+1} = (\mathbf{H} - \mathbf{E}_t)^2 \mathbf{f}_n - a_n \mathbf{f}_n - b_n \mathbf{f}_{n-1} \tag{1}$$

where $\mathbf{H}$ is a given operator, $\mathbf{f}$ is basis functions, a and b are coefficients and n is Lanczos step number. $\mathbf{E}_t$ is trial energy which corresponds to shift in structural dynamics. The concept of power technique in (1) can be applied to the eigenproblem in structural dynamics. The eigenproblem of structure frequently encountered in structural dynamics can be expressed as

$$\mathbf{K}\boldsymbol{\phi}_i = \lambda_i \mathbf{M}\boldsymbol{\phi}_i \quad (i = 1, 2, 3, \cdots, n) \tag{2}$$

where $\mathbf{M}$ and $\mathbf{K}$ are symmetric mass and stiffness matrices of order n, respectively. λ_i and $\boldsymbol{\phi}_i$ are the ith eigenvalue and associated eigenvector of the system. To get the solution of (2), Lanczos schemed Ritz bases vectors through Gram-Schmidt orthogonalization of Krylov sequence as follows:[8]

$$\mathbf{x}_{i+1} = (\mathbf{K}_\mu^{-1}\mathbf{M})^i \mathbf{x}_0 - \sum_{j=1}^{i} \upsilon_j \mathbf{x}_j \tag{3}$$

where $\mathbf{x}_0$ is a starting vector, $\mathbf{x}_j$ is jth Lanczos vector, υ_j is the component of $\mathbf{v}_i$ along $\mathbf{x}_j$, $\mathbf{K}_\mu = \mathbf{K} - \mu\mathbf{M}$ and μ is shift. The concept of power technique can be applied to the dynamic matrix in (3), then following modified Gram-Schmidt orthogonalization can be introduced.

$$\mathbf{x}_{i+1} = ((\mathbf{K}_\mu^{-1}\mathbf{M})^\delta)^i \mathbf{x}_0 - \sum_{j=1}^{i} \upsilon_j \mathbf{x}_j \tag{4}$$

Where δ is positive integer. (4) means that an approximated eigenvector, whose number of iteration is δi, is contained in $(i+1)$ Lanczos vectors. Whereas, in (3), $(i+1)$ Lanczos vectors contain an approximated eigenvector whose number of iterations is i. Therefore, (4) gives a better solution than (3). From (4), modified Lanczos recursion can be derived as

$$\tilde{\mathbf{x}}_i = (\mathbf{K}_\mu^{-1}\mathbf{M})^\delta \mathbf{x}_i - \alpha_i \mathbf{x}_i - \beta_{i-1}\mathbf{x}_{i-1} \tag{5}$$

where α_i and β_i are scalar coefficients obtained by

$$\alpha_i = \mathbf{x}_i^T \mathbf{M} (\mathbf{K}_\mu^{-1}\mathbf{M})^\delta \mathbf{x}_i, \quad \beta_i = (\tilde{\mathbf{x}}_i^T \mathbf{M} \tilde{\mathbf{x}}_i)^{1/2} \tag{6}$$

then the next Lanczos vector is

$$\mathbf{x}_{i+1} = \tilde{\mathbf{x}}_i / \beta_i \tag{7}$$

With a set of Lanczos vectors, $\mathbf{X} = [\mathbf{x}_1\ \mathbf{x}_2 \cdots \mathbf{x}_q]$, we can obtain the tridiagonalized standard eigenproblem of reduced order $q << n$

$$\mathbf{T}\tilde{\phi}_i = (1/(\lambda_i - \mu)^\delta)\tilde{\phi}_i \quad (i = 1,2,3,\cdots,q) \tag{8}$$

where

$$\mathbf{T} = \mathbf{X}^T\mathbf{M}(\mathbf{K}_\mu^{-1}\mathbf{M})^\delta\mathbf{X} = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & & \ddots & & \\ & & & \alpha_{q-1} & \beta_{q-1} \\ & & & \beta_{q-1} & \alpha_q \end{bmatrix} \tag{9}$$

The Lanczos algorithm is subjected to loss of orthogonality of Lanczos vectors due to round-off errors. In this paper, full reorthogonalization process[1] is used to retain the orthogonality of the Lanczos vectors. The number of total operations for the matrix-powered Lanczos algorithm is

$$N_{total} = (1/2)nm^2 + (q^2 + 4q\delta + 5q + 3/2)nm + \{(3/2)q^2 + q\delta + (17/2)q\}n + 10q^2 + q + \sum_{j=2}^{q} 6js_j \tag{10}$$

where n is system order, m half-bandwidth, q the number of calculated Lanczos vectors and s_j the number of iterations of jth step in QR iteration for the eigenvalues of tridiagonal system.

3. Numerical examples

A simple spring-mass system with 100 DOFs[3], a plane framed structure[2], a three-dimensional frame structure[2] and a three-dimensional building frame[9] are analyzed to verify the effectiveness of the matrix-powered Lanczos method. With the predetermined error norm of 10^{-6}, the number of operations for calculating desired eigenpairs is compared. To examine the suitable power of dynamic matrix, numerical examples are analyzed with varying power of dynamic matrix. System matrices of a simple spring-mass system are

$$\mathbf{M} = \mathbf{I},\ \mathbf{K} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \tag{11}$$

The geometric configurations and the material properties of a plane framed structure, a three-dimensional frame structure and a three-dimensional building frame are shown in Figs. 1 ~ 3.

Some results are shown in Table 1 and Fig. 4. The 1st power (δ = 1) corresponds to the conventional Lanczos method. Table 1 and Fig. 4 show that the convergence of the matrix-powered Lanczos method is better than that of the conventional Lanczos method. However, in some cases, high matrix power causes failure in convergence due to the numerical instability.

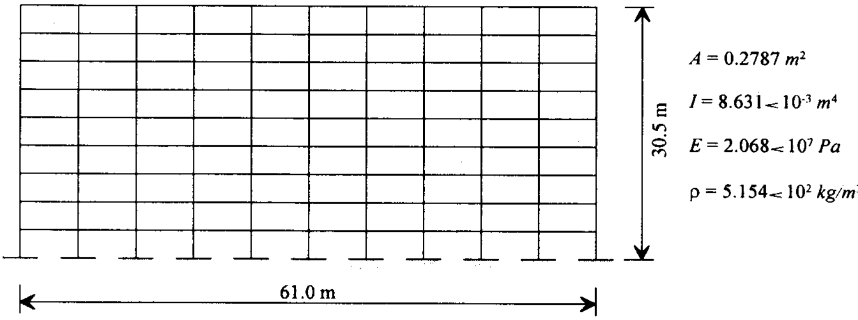
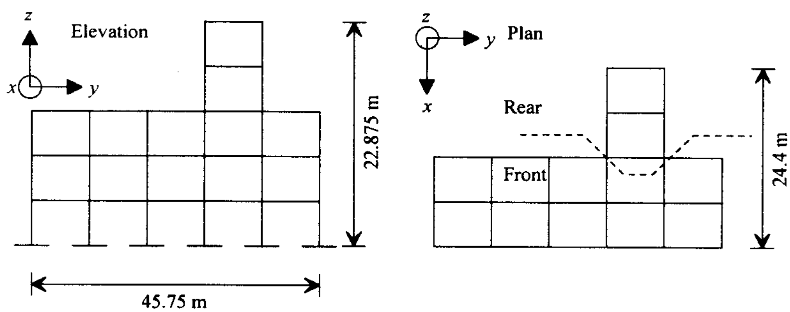


Fig. 1. Plane framed structure (DOFs: 330)



Column in front building : $A = 0.2787\ m^2$, $I = 8.631 \times 10^{-3}\ m^4$
Column in Rear building : $A = 0.3716\ m^2$, $I = 10.789 \times 10^{-3}\ m^4$
All beams into x-Diretion : $A = 0.1858\ m^2$, $I = 6.473 \times 10^{-3}\ m^4$
All beams into y-Diretion : $A = 0.2787\ m^2$, $I = 8.631 \times 10^{-3}\ m^4$

$E = 2.068 \times 10^7\ Pa$
$\rho = 5.154 \times 10^2\ kg/m^3$

Fig. 2. Three-dimensional frame structure (DOFs: 468)

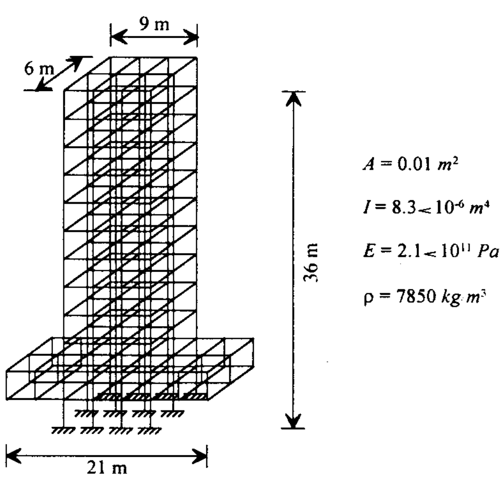


Fig. 3. Three-dimensional building frame (DOFs: 1008)

Table 1. Number of operations for calculating desired eigenpairs

Structure	No. of eigenpairs	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$
Simple spring-mass system	2	38663	29823	26954	23653
	4	78922	58529	47567	44122
	6	120458	85712	73040	69391
	8	157649	117587	103055	99550
	10	214729	154418	138122	*
Plane framed structure	6	10908273	7429050	7072452	6633536
	12	20855865	13578945	11688377	11237625
	18	27029145	18676209	16508507	16047093
	24	31581179	22516533	20164797	*
	30	102944376	65994807	54112986	*
Three-dimensional frame structure	10	71602154	50687925	48705515	46214349
	20	181780512	124269611	116680070	108715163
	30	307269560	215884077	192064376	182518601
	40	684162222	453454527	378770940	356596304
	50	1024104917	656188310	553972908	504420108
Three-dimensional building frame	20	395079020	278717178	*	*
	40	1196316954	801878160	*	*
	60	3045578295	1993108128	*	*
	80	3398746793	2509125474	*	*
	100	3536190824	3625240574	*	*

* convergence failure

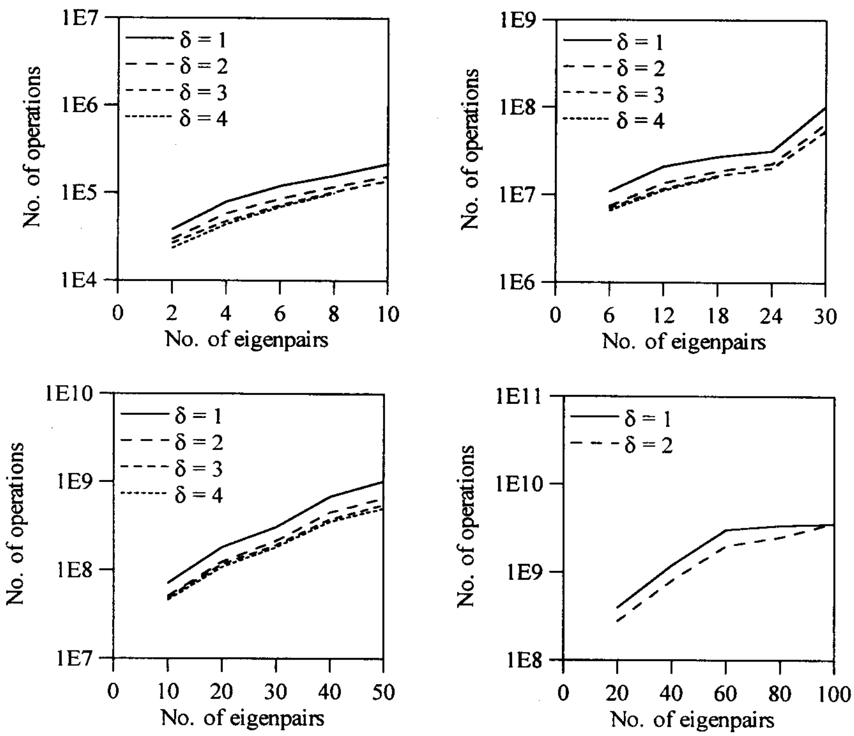


Fig. 4. Number of operations for calculating desired eigenpairs

4. Conclusions

This paper investigates the applicability of the matrix-powered Lanczos method to the eigenproblem of structures. The characteristics of the matrix-powered Lanczos method by the numerical results from examples are summarized as follows:

(1) Since the power of the dynamic matrix can reduce the required number of Lanczos vectors, the matrix-powered Lanczos method has not only the better convergence but also the less operation count than the conventional Lanczos method

(2) The suitable power of the dynamic matrix that gives numerically stable solution in the matrix-powered Lanczos method is the second power.

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