

# 고유진동수와 모드형의 민감도를 구하기 위한 반복법

## An Iterative Method for Natural Frequency and Mode Shape Sensitivities

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ABSTRACT: A numerical method is presented for computation of eigenvector derivatives used an iterative procedure with guaranteed convergence. An approach for treating the singularity in calculating the eigenvector derivatives is presented, in which a shift in each eigenvalue is introduced to avoid the singularity. If the shift is selected properly, the proposed method can give very satisfactory results after only one iteration. A criterion for choosing an adequate shift, dependent on computer hardware is suggested: it is directly dependent on the eigenvalue magnitudes and the number of bits per numeral of the computer. Another merit of this method is that eigenvector derivatives with repeated eigenvalues can be easily obtained if the new eigenvectors are calculated. These new eigenvectors lie "adjacent" to the  $m$  (number of repeated eigenvalues) distinct eigenvectors, which appear when the design parameter varies. As an example to demonstrate the efficiency of the proposed method in the case of distinct eigenvalues, a cantilever plate is considered. The results are compared with those of Nelson's method which can find the exact eigenvector derivatives. For the case of repeated eigenvalues, a cantilever beam is considered. The results are compared with those of Dailey's method which also can find the exact eigenvector derivatives. The design parameter of the cantilever plate is its thickness, and that of the cantilever beam its height.

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## 1. INTRODUCTION

Eigenderivatives are extremely useful for determining the sensitivity of dynamic responses to system parameter variations. Knowledge of the eigenvector derivatives with respect to physical parameters can help an engineer optimize a structural design or minimize its sensitivity to the parameters. Such information can be used regularly for structural design and optimization, and for the improvement of the agreement between analytical and experimental results[1]. For structural control systems, eigenderivatives can be directly applied to system identification and robust performance tests. For structural design, eigenderivatives can be used to optimize the natural frequencies and mode shapes of a structure by varying its design parameters.

In 1976, Nelson[2] presented an algorithm for computing the eigenvalue and eigenvector derivatives of general real matrices with non-repeated eigenvalues. Nelson's method was a significant advance because it required only the knowledge of the eigenvectors to be differentiated. Previous methods required computing all or most of the eigenvectors, which consumed too much computational time for the high order matrices encountered in modern finite element analyses. Nelson's method, however, suffers from singularity problems when repeated eigenvalues exist. In typical structures, there are many repeated or nearly equal eigenvalues, due to structural symmetry. A number of papers[3~12] have been presented to find the eigenvector derivatives in the case of repeated eigenvalues. In 1988, Dailey[10] presented an

efficient method which amended Ojalvo's method[6] for computing eigenvalue and eigenvector derivatives of real symmetric matrices with repeated eigenvalues. However, Dailey's method is complicated for finding eigenvector derivatives and clumsy for programming because it basically follows Nelson's algorithm. Besides the above methods, there are methods in which an iteration scheme is used: the iterative method proposed by Rudisill and Chu[13] in 1975, later refined by Andrew[14] in 1978, and with accelerated convergence by Tan[15] in 1986. These methods have very complicated algorithms and need several iterations to obtain satisfactory accuracy.

The method to be proposed in what follows here has a guaranteed convergence, has a very simple and compact algorithm, and needs only one iteration to obtain satisfactory results if the shift is chosen properly. It is natural that one needs to take the shift properly since the eigenvalues are known from the previous frequency analysis. Furthermore, the iterative procedure of the proposed method is very useful for repeated eigenvalues as well as non-repeated ones if new eigenvectors are calculated which lie "adjacent" to the  $m$  (number of repeated eigenvalues) distinct eigenvectors, which appear when the design parameter varies. The key point of the present method is to determine the shift properly, and this is dependent on the eigenvalue magnitudes and the number of decimal points of the computer.

In the first section of this paper both Nelson's method for finding eigenvalue and eigenvector derivatives in the case of

non-repeated eigenvalues, and Dailey's method for repeated eigenvalues are reviewed, as these methods have hitherto been acknowledged to be efficient. In the second section the proposed method is applied to cases of non-repeated and repeated eigenvalues, and its convergence is established. Numerical examples are presented in section 4.

## 2. CURRENT METHODS FOR SENSITIVITY ANALYSIS

### 2.1 Technical background

The real symmetric eigenvalue problem associated with linear vibration is defined as

$$K\phi_j = \lambda_j M\phi_j \quad (1)$$

where  $K$  and  $M$  are the stiffness and mass matrices; they are order  $n$  symmetric matrices.  $M$  is positive definite and  $K$  is positive definite or semi-positive definite.  $\phi_j$  is the  $j$ th mode shape,  $\lambda_j$  is the square of the  $j$ th natural frequency. The eigenvector  $\phi_j$  is typically normalized as

$$\phi_j^T M \phi_j = 1 \quad (2)$$

To obtain the derivative of the eigenvalue, Eq. (1) is differentiated with respect to a design parameter,

$$(K - \lambda_j M)\phi_j' = -(K' - \lambda_j M')\phi_j + \lambda_j' M \phi_j \quad (3)$$

Premultiplying each side of Eq. (3) by  $\phi_j^T$  gives the eigenvalue derivative as

$$\lambda_j' = \phi_j^T (K' - \lambda_j M') \phi_j \quad (4)$$

where  $\lambda_j' = \partial \lambda_j / \partial p$ ,  $K' = \partial K / \partial p$  and  $M' = \partial M / \partial p$ , where  $p$  is the design parameter. Thus, the right side of Eq. (3) is known but the eigenvector derivative  $\phi_j'$  cannot be found directly since the matrix  $[K - \lambda_j M]$  is singular. To find the eigenvector derivative, Nelson[2] proposed an algorithm expressing the eigenvector derivative in terms of a particular solution and a complementary solution. Also there are many methods in which the modal concept is used[3, 7~9, 16]. The analysis time of these methods is longer than that of Nelson's method, as shown in Ref. [17], since they need all or most of the eigenvectors to find the eigenvector derivative.

### 2.2 Nelson's method

In the method, one expresses the eigenvector derivative in terms of a particular solution  $v_j$  and a homogeneous solution  $c_j \phi_j$ :

$$\phi_j' = v_j + c_j \phi_j \quad (5)$$

where  $c_j$  is an undetermined coefficient. The particular solution is found by identifying the component of the  $j$ th eigenvector with the largest absolute value and constraining its derivative to zero. The undetermined coefficient can be obtained by substituting Eq. (5) into the derivative of Eq. (2),

$$c_j = -v_j^T M \phi_j - 0.5 \phi_j^T M' \phi_j \quad (6)$$

Nelson's method is a powerful one for computing the eigenvector derivatives of general real matrices with distinct eigenvalues because it requires knowledge of only the eigenpairs that are to be differentiated. However, the algor-

ithm of the method is lengthy and complicated. The complete procedure of Nelson's method is summarized in Table 1.

Table 1. The procedure of Nelson's method

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- (1) Compute  $\lambda_j' = \phi_j^T (K' - \lambda_j M') \phi_j$ .
  - (2) Let  $f_j = -(K' - \lambda_j M') \phi_j + \lambda_j M \phi_j$ , and  $G_j \equiv K - \lambda_j M$ .
  - (3) Find  $k$  such that  $\phi_{jk}$  is the largest element on the column of  $\phi_j$ , where  $\phi_j = (\phi_{j1}, \phi_{j2}, \dots, \phi_{jn})^T$ .
  - (4) Construct  $\bar{G}_j$  by zeroing out row  $k$  and column  $k$  of  $G_j$  and setting the  $k$ th diagonal element to 1.
  - (5) Construct  $\bar{f}_j$  by zeroing out the element of  $f_j$ .
  - (6) Solve  $\bar{G}_j v_j = \bar{f}_j$ .
  - (7) Compute  $c_j = -v_j^T M \phi_j - 0.5 \phi_j^T M' \phi_j$ .
  - (8) Let  $\phi_j' = v_j + C_j \phi_j$ .
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### 2.3 Dailey's method : case of repeated eigenvalues

The algorithm is an extension of Ojalvo's work[6] by Dailey[10]. When the eigenvalues are multiple and a design parameter is perturbed, the eigenvectors split into as many as  $m$  (multiplicity of multiple eigenvalues) distinct eigenvectors. One seeks the derivatives of these distinct eigenvectors which appear with design parameter perturbation. For the derivatives of the eigenvectors to exist, the eigenvectors must be adjacent to the  $m$  distinct eigenvectors that appear when the design parameter varies. Otherwise, the eigenvectors would jump discontinuously with varying design parameter. Consider the following eigenvalue problem, in which  $\Psi$ , of order  $(n \times m)$  is a matrix of eigenvectors with multiple eigenvalues :

$$K\Psi = M\Psi\Lambda, \quad (7)$$

Here  $\Lambda = \lambda I$  and  $\Psi^T M \Psi = I$ , and  $\lambda$  is the

eigenvalue for the eigenspace spanned by the columns of  $\Psi$ ;  $\lambda$  is an eigenvalue of multiplicity  $m$ . Adjacent eigenvectors can be expressed in terms of  $\Psi$  by an orthogonal transformation such as

$$Z = \Psi\Gamma, \quad (8)$$

where  $\Gamma$  is an orthonormal transformation matrix and its order is  $(m \times m)$ ; also  $\Gamma^T \Gamma = I$ . The columns of  $Z$  are the adjacent eigenvectors for which a derivative can be defined. The adjacent eigenvectors satisfy the  $M$ -orthogonality condition.

$$Z^T M Z = \Gamma^T \Psi^T M \Psi \Gamma = \Gamma^T \Gamma = I. \quad (9)$$

The next step is to find  $\Gamma$  such that the derivative of eigenvectors exists and then to find  $Z$ ,  $\Lambda'$  and  $Z'$ .  $\Lambda'$  is obtained as follows :

$$\Lambda' = \partial \Lambda / \partial p = \text{diag} (\lambda_1, \dots, \lambda_m) \quad (10)$$

Consider the following eigenvalue problem to find  $Z$ ,  $\Lambda'$  and  $Z'$ .

$$KZ = MZA \quad (11)$$

Differentiating Eq.(11) with respect to a design parameter and rearranging it yield,

$$(K - \lambda M)Z' = -(K' - \lambda M')Z + MZA' \quad (12)$$

Premultiplying by  $\Psi^T$  and substituting  $Z = \Psi\Gamma$  into Eq.(12) yield,

$$[\Psi^T (K' - \lambda M') \Psi] \Gamma \equiv D\Gamma = \Gamma \Lambda' \quad (13)$$

One obtains the eigenvalue derivative  $\Lambda'$  and the orthogonal transformation matrix  $\Gamma$  by

solving Eq.(13), and then the adjacent eigenvectors  $Z$  by the relation  $Z=\Psi\Gamma$ .

To find  $Z'$ , returning to Eq.(12), and defining  $F\equiv(\lambda M'-K')Z+MZA'$  yields,

$$(K-\lambda M)Z'=F \quad (14)$$

A direct solution of Eq.(14) is not possible since the matrix  $[K-\lambda M]$  is singular. The matrix  $[K-\lambda M]$  has rank $(n-m)$  and a null space spanned by the eigenvectors with multiple eigenvalues, if  $V$  is any solution of  $(K-\lambda M)V=F$  then  $V+ZC$  is also a solution, where  $C$  is the  $(m\times m)$  coefficient matrix. First, consider a procedure proposed by Nelson to find the particular solution  $V$ . The procedure for finding  $V$  is summarized in Table 2.

Table 2. The procedure to find the particular solution  $V$

- (1) Let  $Z=[z_{ij}]$
- (2) Find  $k$  such that  $z_{1,k}$  is the largest element on the first column of  $Z$ .
- (3) Replace the  $k$ th row and column of  $G$  with zeros, except placing 1 on the  $k$ th diagonal element.
- (4) Replace the  $k$ th row of  $F$  with zeros.
- (5) Go back to step 2 and repeat for the next column of  $Z$  until through. If  $k$  has been used before, choose the second largest (third largest, etc.) element in the column instead.
- (6) Call the resulting matrices  $\bar{G}$  and  $\bar{F}$ .
- (7) Solve  $\bar{G}V=\bar{F}$

The diagonal elements of  $C$  can be obtained by substituting  $V+ZC$  into the derivative of the normalization condition  $Z^T MZ=I$ . The resulting equation is

$$C+C^T=-V^T MZ-Z^T M V-Z^T M'Z\equiv Q, \quad (15)$$

that is,

$$c_{ii}=0.5 q_{ii} \quad (16)$$

The off-diagonal elements of  $C$  can be determined by differentiating the eigenvalue problem  $KZ=MZA$  twice. Differentiating Eq.(11) with respect to the design parameter gives

$$K'Z+KZ'M'Z\Lambda-MZ'\Lambda-MZA'=0. \quad (17)$$

Differentiating Eq.(17) with respect to the design parameter, and rearranging yield

$$(K''-\lambda M'')Z+2(K'-\lambda M')Z'+(K-\lambda M)Z''-2M'ZA''-2MZ'\Lambda''-MZA''=0. \quad (18)$$

Premultiplying Eq.(18) by  $Z^T$  and expanding it with  $Z'=V+ZC$  yields

$$Z^T(K''-\lambda M'')Z+2Z^T(K'-\lambda M')V+2Z^T(K'-\lambda M')ZC-2Z^T M'ZA''-2Z^T MVA''-2Z^T MZCA''-\Lambda''=0 \quad (19)$$

Using the relations  $Z^T(K'-\lambda M')Z=Z^T[MZA'-(K-\lambda M)]Z'=\Lambda'$  and  $Z^T MZ=I$ , then gives

$$CA'-\Lambda'C+0.5\Lambda''=Z^T(K'-\lambda M')V-Z^T(M'Z+MV)\Lambda''+0.5Z^T(K''-\lambda M'')Z\equiv R. \quad (20)$$

Note that  $\Lambda''$  is diagonal, whereas  $CA'-\Lambda'C$  always has zeros on the diagonal. This provides a neat separation of  $C$  and  $\Lambda''$  and allows solving for both from this one equation. If  $C=[c_{ij}]$ ,  $R=[r_{ij}]$ ,  $\Lambda''=\partial\Lambda/\partial p=\text{diag}(\lambda_1, \dots, \lambda_m)$  and  $\Lambda''=\text{diag}(\lambda_1, \dots, \lambda_m)$ , it is easily seen that

$$r_{ij} = \left\{ \begin{array}{ll} c_{ij}(\lambda_j - \lambda_i), & \text{if } j \neq i \\ 0.5\lambda_i, & \text{otherwise} \end{array} \right\} \quad (21)$$

Therefore,  $c_{ij} = r_{ij} / (\lambda_j' - \lambda_i')$  if  $\lambda_j' \neq \lambda_i'$  and  $c_{ii} = 0.5 q_{ii}$  in Eq.(16). When multiple eigenvalue derivatives exist, that is  $\lambda_j' = \lambda_j'(j \neq i)$ , the original m-di-mensional eigenspace is not separating into m distinct one-dimensional eigenspaces as the design parameter varies. In this case, any corresponding column  $f$   $Z' = V - ZC$  is a valid eigenvector derivative as shown in Refs.[3, 10] if the non-unique elements of C satisfy the constraint  $c_{ij} - c_{ji} = q_{ij} = q_{ji}$ . The eigenvector derivative is not unique. The complete algorithm of Dailey's method is summarized in Table 3.

Table 3. The algorithm of Dailey's method.

- (1) Compute  $D = \Psi^T(K' - \lambda M')\Psi$ .
- (2) Solve the eigenvalue problem  $D\Gamma = \Gamma\Lambda'$ .  $\Lambda'$  is the diagonal matrix of eigenvalue derivatives, and  $\Gamma$  should be normalized so that  $\Gamma^T\Gamma = I$ .
- (3) Let the columns of  $Z = \Psi\Gamma$  be the new eigenvectors.
- (4) Compute  $G = K - \lambda M$ ,  $F = (\lambda M' - K')Z + MZA'$ .
- (5) Find the m rows of Z containing the largest elements. Zero out these rows and columns of G and the same rows of F. Place 1 in the affected diagonal elements of G and call the resulting matrices  $\bar{G}$  and  $\bar{F}$ .
- (6) Solve  $\bar{G}V = \bar{F}$ .
- (7) Compute  $Q = C + C^T = -V^TMZ - Z^TMV - Z^TM'Z$ .
- (8) Compute  $R = CA' - \Lambda'C + 0.5\Lambda' = Z^T(K' - \lambda M')V - Z^T(M'Z + MV)\Lambda' + 0.5Z^T(K' - \lambda M')Z$ .
- (9) Construct the  $m \times m$  matrix C by the rule

$$c_{ij} = \begin{cases} r_{ij} / (\lambda_j' - \lambda_i') & \text{if } \lambda_j' \neq \lambda_i' \\ 0.5 q_{ij} & \text{otherwise} \end{cases}$$

where  $\Lambda' = \text{diag}(\lambda_1', \dots, \lambda_m')$ .

- (10) Let  $Z' = V + ZC$ . The columns of  $Z'$  are the eigenvector derivatives.

### 3. PROPOSED METHOD

#### 3.1 Case of non-repeated eigenvalues

Consider Eq.(3), and define  $f_j \equiv -(K' - \lambda_j$

$$M')\phi_j + \lambda_j M\phi_j$$

$$(K - \lambda_j M)\phi_j = f_j \quad (22)$$

For a shift  $\Delta\lambda_j$  in Eq.(22), with the definition

$$A_N \equiv [K - (\lambda_j - \Delta\lambda_j)M],$$

$$\begin{aligned} [K - (\lambda_j - \Delta\lambda_j)]\phi_j &= f_j + \Delta\lambda_j M\phi_j \text{ or } A_N\phi_j \\ &= f_j + \Delta\lambda_j M\phi_j \end{aligned} \quad (23)$$

The proposed iteration scheme is as follows. First

$$\tilde{\phi}^{(k+1)} = A_N^{-1} [f_j + \Delta\lambda_j M\phi^{(k)}], \quad (24)$$

where  $\tilde{\phi}^{(k+1)}$  can be computed, because  $A_N$  is nonsingular. The starting vector can be chosen arbitrarily. ( $\tilde{\phi}^{(k+1)}$  must be modified at the end of the iteration by Eq.(26), which follows.) The derivative of the eigenvector such that Eq.(2) is to be satisfied can be obtained by substituting  $\tilde{\phi}^{(k+1)}$  into the derivative of Eq.(2). That is, differentiating Eq.(2),

$$2\phi_j^T M\phi_j' + \phi_j^T M'\phi_j = 0, \quad (25)$$

and substituting  $\tilde{\phi}^{(k+1)} = d_j^{(k+1)}\tilde{\phi}^{(k+1)}$  into Eq.(25), where  $d_j^{(k+1)}$  is a coefficient to be determined, give

$$\phi_j^{(k+1)} = -\frac{\phi_j^T M'\phi_j}{2\phi_j^T M\tilde{\phi}^{(k+1)}} \tilde{\phi}^{(k+1)} \quad (26)$$

One can obtain the satisfactory results after only one iteration if the shift is chosen as small as possible. For computer operation, however, the shift must be larger than a critical value at which the inverse of  $A_N$  can exist in the com-

puter. If the shift is smaller than the critical value,  $A_N$  is still almost singular. The lower bound criterion for the allowed shift  $\Delta\lambda_j$  is

$$10^{-N} < \Delta\lambda_j / \lambda_j, \quad (27)$$

where  $N$  is the number of bits per numeral of the computer. If  $\lambda_j=0$ , the shift can be taken as  $\Delta\lambda_j > 10^{-N}$ . The upper bound of the shift will be discussed in the convergence analysis (see section 3.3).

The algorithm of the proposed method in the case of non-repeated eigenvalue is summarized in Table 4.

Table 4. The algorithm of the proposed method in the case of non-repeated eigenvalues

- (1) Compute  $\lambda_j = \phi_j^T (K' - \lambda_j M') \phi_j$ .
- (2) Let  $f_j = -(K' - \lambda_j M') \phi_j + \lambda_j M \phi_j$ , and  $A_N \equiv [K - (\lambda_j - \Delta\lambda_j) M]$ .
- (3) Choose arbitrary starting vector  $\phi_j^{(0)}$ , and compute  $\tilde{\phi}^{(k+1)} = A_N^{-1} [f + \Delta\lambda M \phi^{(k)}]$ .
- (4) Iterate step (3).
- (5) Compute  $\phi_j^{(k+1)} = \frac{\phi_j^T M' \phi_j}{2\phi_j^T M \phi_j^{(k+1)}} \tilde{\phi}^{(k+1)}$ .

Note that the proposed method, as well as Nelson's method, has the desirable properties of preserving the symmetries of the stiffness and mass matrices, and of requiring knowledge of only one eigenpair. Both properties are important in realistic structural problems where the stiffness and mass matrices are of very high order.

### 3.2 Case of repeated eigenvalue

The adjacent eigenvectors must be calculated first to apply the proposed method in the case of repeated eigenvalues. This

procedure is the same as that of Dailey's method. If the adjacent eigenvectors are calculated, the proposed method can be applied immediately in the case of repeated eigenvalues. Consider the following eigenvalue problem with the normalization condition  $Z^T M Z = I$ .

$$K Z = M Z \Lambda. \quad (28)$$

Here  $\Lambda = \text{diag}(\lambda_j, \lambda_{j+1}, \dots, \lambda_{j+m-1})$  and  $\lambda_j = \lambda_{j+1} = \dots = \lambda_{j+m-1}$  ( $=\lambda$ ).

$$\text{Rewriting Eq. (14), } (K - \lambda M) Z' = F \quad (14)$$

where  $Z' = [z'_1, \dots, z'_m]^T$  and  $F = [f_1, \dots, f_m]^T$  and considering a shift  $\Delta\lambda$  in Eq. (14), with the definition  $A_R \equiv [K - (\lambda - \Delta\lambda) M]$ , one has

$$A_R Z' = F + \Delta\lambda M Z'. \quad (29)$$

The proposed iteration scheme for repeated eigenvalues is then :

$$\tilde{Z}^{(k+1)} = A_R^{-1} [F + \Delta\lambda M Z^{(k)}]. \quad (30)$$

This iteration scheme can be used directly to find  $Z'$  in Eq. (14).

The algorithm of the proposed method in the case of repeated eigenvalues is summarized in Table 5.

The proposed algorithm in the case of non-repeated eigenvalues can be applied to Eq. (14) for each column of  $Z'$  and  $F$  independently. That is,  $z'_i$  can be calculated by using the proposed algorithm in the case of non-repeated eigenvalues if the right side of Eq. (14),  $f_i$ , is determined once.

Table 5. Algorithm of the proposed method in the case of repeated eigenvalues

- (1) Compute  $D = \Psi^T [K' + \lambda M'] \Psi$
- (2) Solve the eigenvalue problem  $D\Gamma = \Gamma\Lambda'$ .  $\Lambda'$  is the diagonal matrix of eigenvalue derivatives, and  $\Gamma$  should be normalized so that  $\Gamma^T \Gamma = I$ .
- (3) Let the columns of  $Z = \Psi\Gamma$  be the new eigenvectors.
- (4) Compute  $F \equiv (\lambda M' - K')Z + MZ\Lambda'$ , and let  $A_R \equiv [K - (\lambda - \Delta\lambda)M]$
- (5) Choose an arbitrary starting matrix  $Z^{(0)}$ , and compute  $\tilde{Z}^{(k+1)} = A_R^{-1} [F + \Delta\lambda_j MZ^{(k)}]$
- (6) Iterate step (5).
- (7) Compute  $z_i^{(k+1)} = -\frac{\tilde{z}_i^T M' Z_j}{2\tilde{z}_i^T M' \tilde{Z}_i^{(k+1)}} \tilde{z}_i^{(k+1)}$ , where  $i=1, \dots, m$

It thus has been shown that the present method can be applied to repeated eigenvalue problems as well as to non-repeated ones. The second order derivatives of the stiffness and mass matrices and the complicated procedure for determining the coefficient matrix C encountered in Dailey's method are not needed in the proposed method. Comparatively, this reduces the computer space by about 33% (Dailey's method needs computer space for K, M, K', M' and K'', M'', whereas in the proposed method space for only K, M and K', M' is needed).

### 3.3 Convergence analysis

To analyze the convergence properties of the iteration procedure, one can use the concept of vector iteration. Eq.(23) is rewritten as

$$[K - (\lambda_j - \Delta\lambda_j)M] \phi_j' = f_j + \Delta\lambda_j M \phi_j \quad (31)$$

The first step is to consider the problem in the basis of eigenvectors  $\Phi$ . Defining  $\lambda_j' = (\lambda_j - \Delta\lambda_j)$

and using the transformation  $\phi' = \Phi\phi$ , where  $\Phi$  is the modal matrix and  $\phi$  is the modal coordinate, one obtains the following equivalent eigenproblem :

$$[\Lambda - \lambda_j' I] \phi = \Phi^T f_j - \Delta\lambda_j \phi \quad (32)$$

The proposed iteration scheme is

$$[\Lambda - \lambda_j' I] \phi^{(k+1)} = \Phi^T f_j + \Delta\lambda_j \phi^{(k)} \quad (33)$$

Assume that the arbitrary starting vector is  $\phi^{(0)}$ , and that all eigenvalues are distinct. Consider the iteration process.

$$\begin{aligned} \phi^{(1)} &= [\Lambda - \lambda_j' I]^{-1} [\Delta\lambda_j \phi^{(0)} + \Phi^T f_j] \\ &= \Delta\lambda_j [\Lambda - \lambda_j' I]^{-1} \phi^{(0)} + [\Lambda - \lambda_j' I]^{-1} \Phi^T f_j \\ \phi^{(2)} &= [\Lambda - \lambda_j' I]^{-1} \{ \Delta\lambda_j^2 [\Lambda - \lambda_j' I]^{-1} \phi^{(0)} \\ &\quad + \Delta\lambda_j [\Lambda - \lambda_j' I]^{-1} \Phi^T f_j + \Phi^T f_j \} \\ &= [\Delta\lambda_j^2 [\Lambda - \lambda_j' I]^{-2} \phi^{(0)} + \{ [\Lambda - \lambda_j' I]^{-1} + \Delta\lambda_j \\ &\quad [\Lambda - \lambda_j' I]^{-2} \Phi^T f_j \\ \dots &\quad \dots \quad \dots \quad \dots \\ \phi^{(k)} &= [\Delta\lambda_j^k [\Lambda - \lambda_j' I]^{-k} \phi^{(0)} \\ &\quad + \{ [\Lambda - \lambda_j' I]^{-1} + \Delta\lambda_j [\Lambda - \lambda_j' I]^{-2} + \dots + \Delta\lambda_j^{k-1} \\ &\quad [\Lambda - \lambda_j' I]^{-k} \} \Phi^T f_j \end{aligned} \quad (34)$$

Thus, if all diagonal elements of the matrix  $[\Lambda - \lambda_j' I]$  are larger than  $\Delta\lambda_j$ ,  $\phi^{(k)}$  will converge. That is, the convergence condition is

$$\left| \frac{\Delta\lambda_j}{\lambda_i - \lambda_j'} \right| < 1 \text{ or } \left| \frac{\Delta\lambda_j}{(\lambda_i + \lambda_j) + \Delta\lambda_j} \right| < 1, \text{ for } i=1, 2, \dots, n (i \neq j). \quad (35)$$

For  $\lambda_1 < \lambda_2 < \dots < \lambda_n$ , the upper bound of the convergence condition is



$$\Delta\lambda_j < \frac{\lambda_j - \lambda_{j-1}}{2} \quad (36)$$

The convergence rate in the iteration is determined by the element  $\Delta\lambda_j / [(\lambda_i - \lambda_j) + \Delta\lambda_j]$ , which is largest in absolute magnitude,  $i \neq j$ ; i.e., the convergence rate  $r$  is

$$r = \max_{i \neq j} \left| \frac{\Delta\lambda_j}{(\lambda_i - \lambda_j) + \Delta\lambda_j} \right| \quad (37)$$

The convergence rate of the iteration process is either  $\Delta\lambda_j / [(\lambda_{j-1} - \lambda_j) + \Delta\lambda_j]$  or  $\Delta\lambda_j / [(\lambda_{j+1} - \lambda_j) + \Delta\lambda_j]$ , whichever is larger.

The proposed iteration method is similar to the inverse vector iteration method with shifting [18]. The convergence rate is as high if  $\Delta\lambda_j$  can be chosen as small as possible. Note that one can select the proper shift since the eigenvalues are known by the previous frequency analysis. The lower bound criterion is derived from the critical value for the singularity and the upper bound from the stability for the convergence. Hence, the lower and upper criteria, which are dependent on the decimal points of the computer and the eigenvalue magnitudes, are

$$10^{-N}(\lambda_j) < \Delta\lambda_j < \frac{\lambda_j - \lambda_{j-1}}{2} \text{ or } 10^{-N} < \frac{\Delta\lambda_j}{\lambda_j} < \frac{1}{2} \left( 1 - \frac{\lambda_{j-1}}{\lambda_j} \right), \quad (38)$$

where  $N$  is the number of bits per numeral of the computer. If  $\lambda_j = 0$ , the shift can be taken as  $\Delta\lambda_j > 10^{-N}$ . If the right side of Eq.(38) is almost equals to zero, that is,  $\lambda_j - \lambda_{j-1} \approx 0$ , the convergence analysis can be applied to the case of the eigenvalue problem with repeated

eigenvalues. If  $\lambda_j = \lambda_{j+1} = \dots = \lambda_{j+m-1}$ , the convergence rate in the case of repeated eigenvalues is

$$r = \max_{i \neq j, j+1, \dots, j+m-1} \left| \frac{\Delta\lambda_j}{(\lambda_i - \lambda_j) + \Delta\lambda_j} \right| \quad (39)$$

and convergence occurs to a vector in the subspace corresponding to  $\lambda_j$ . Even though the system has repeated eigenvalues, the eigenvector derivatives can be obtained after only one iteration, as will be shown in section 4.2.

## 4. NUMERICAL EXAMPLES

### 4.1 Cantilever plate : case of non-repeated eigenvalues

As a simple application, the cantilever plate used in Ref.[11] is considered. It is modeled with 36 triangular elements as shown in Figure 1. Each node has three-degrees-of-freedom element, (z-translation, x-rotation and y-rotation), thus each element has nine-degrees-of-freedom. The number of nodes is 28, and the total degrees-of-freedom of this structure 72. For example calculations, Young's modulus is  $10.5 \times 10^5$  N/m<sup>2</sup>, the mass density  $5.88 \times 10^{-3}$  kg/m<sup>3</sup> and Poisson's ratio 0.3. The lowest ten frequencies of the cantilever plate are given in Table 6. The length of the plate is 6 m, and width 3m and thickness 0.01 m. The design parameter is the plate thickness  $t$ . The stiffness and mass matrices of the structure are proportional to  $t^3$  and  $t$ , respectively. The derivatives of the stiffness and mass matrices can be immediately obtained by differentiating them with respect

to t.

The results for the first eigenvector derivative corresponding to a shift  $\Delta\lambda_1/\lambda_1$  are shown in Figures 2 and 3. The error norm  $\varepsilon$  is defined as

$$\varepsilon = \frac{\|\phi'_{Nelson} - \phi'_{approx}\|_2}{\|\phi'_{Nelson}\|_2} \quad (40)$$

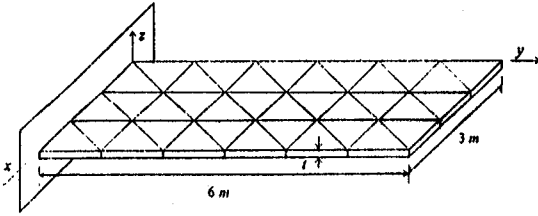


Figure 1. Cantilever plate with the thickness  $t$  as the design parameter. Number of nodes 28, number of elements 36, number of degrees of freedom 72, Young's modulus  $E=10.5 \times 10^5 \text{ N/m}^2$ , mass density  $\rho=5.88 \times 10^{-3} \text{ kg/m}^3$ .

Table 6. The first four natural frequencies of the cantilever plate.

Number	Eigenvalue	Natural frequency(Hz)
1	29.73	0.8678
2	472.42	3.4593
3	1095.04	5.2667
4	4689.00	10.898

Figure 2 shows the error norm for a shift  $\Delta\lambda_1/\lambda_1$  after one iteration, and Figures 3(a)~(d) show the ratio of each element of the first approximate and exact eigenvector derivatives, Rat, according to the number of iterations for various shifts :

$$\text{Rat} \Big|_i = \frac{\phi'_{i, approx}}{\phi'_{i, Nelson}}, \text{ where } i=1, \dots, n \quad (41)$$

All runs were executed in the IRIS4D-20-S17 which has 16 bits per numeral, 10 Mips and 0.9

MFlops. The number of iterations according to the shift which is required to satisfy the predetermined error norm  $\varepsilon_0$  is given in Table 7. The starting vector  $\phi^{(0)}$  is arbitrary, thus zero vector is used in the example. For example, if  $\Delta\lambda_1/\lambda_1$  is smaller than  $10^{-6}$ , the error norm is smaller than  $8.16 \times 10^{-9}$  and Rat is almost 1.0 after one iteration. If a shift is taken  $10^{-16} < \Delta\lambda/\lambda < 10^{-6}$ , nearly exact results can be obtained after only one iteration (see the Figures 3(c), (d)).

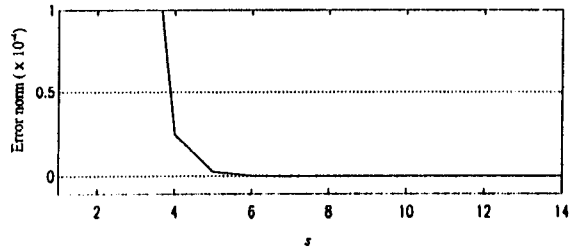


Figure 2. Error norm of the first eigenvector derivative according to the shift after one iteration. Shift  $\Delta\lambda_1/\lambda_1 = 10^{-5}$ .

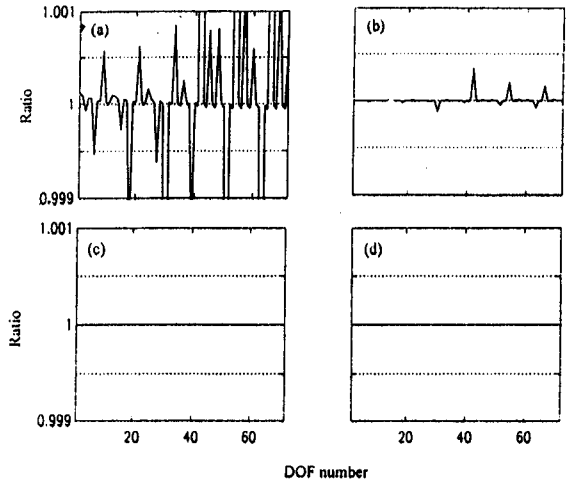


Figure 3. Ratio of each element of the approximate and exact eigenvector derivative after one iteration. ( $\lambda_1=29.7335$ ). (a) When the shift  $\Delta\lambda_1/\lambda_1 = 10^{-4}$ ; (b)  $10^{-6}$ ; (c)  $10^{-8}$ ; (d)  $10^{-10}$ .

Table 7. The number of iterations required to satisfy  $\epsilon_0$  for the cantilever plate.

Tolerance $\epsilon_0$	Shift ( $\Delta\lambda_1/\lambda_1$ ), $\lambda_1=29.7335$												
	$10^{-1}$	$10^{-2}$	$10^{-3}$	$10^{-4}$	$10^{-5}$	$10^{-6}$	$10^{-7}$	$10^{-8}$	$10^{-9}$	$10^{-10}$	$10^{-11}$	$10^{-13}$	$10^{-14}$
$10^{-6}$	3	2	2	2	1	1	1	1	1	1	1	1	1
$10^{-9}$	3	3	3	2	2	2	2	2	1	1	1	1	1
$10^{-12}$	6	5	4	3	2	2	2	2	2	2	1	1	1

#### 4.2 Cantilever beam : case of repeated eigenvalues

Consider as an illustrative example of the case of multiple natural frequencies, the cantilever beam with square section used in Ref.[12] and shown in Figure 4. Each member is modeled as a beam element of which each node has four-degrees-of-freedom(y-translation, z-translation, y-rotation and z-rotation). The number of nodes is 21, and the number of elements 20. Each element has eight-degrees-of-freedom, and the structure has 80 degrees-of-freedom, so the eigenvalue problem is of dimension  $80 \times 80$ . Young's modulus is  $2.10 \times 10^{11}$  N/m<sup>2</sup>, mass density  $7.85 \times 10^3$  kg/m<sup>3</sup>, and Poisson's ratio 0.3. Both the beam height and width are 0.1m, and its length 10m. The lowest ten frequencies of the cantilever beam are given in Table 8, where the first and second are the same, the third and fourth, and so on. The design parameter is the beam height h.

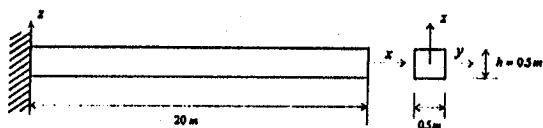


Figure 4. Cantilever beam with the height h as the design parameter. Number of nodes 21, number of elements 20, number of degrees of freedom 80, Young's modulus  $E=2.10 \times 10^{11}$  N/m<sup>2</sup>, mass density  $\rho=7.85 \times 10^3$  kg/m<sup>3</sup>.

Table 8. The first four natural frequencies of the cantilever beam.

Number	Eigenvalue	Natural frequency(Hz)
1	42.98	1.043
2	42.98	1.043
3	1685.66	6.534
4	1685.66	6.534

The results for the first and second eigenvector derivatives are shown in Figures 5~8. Figure 5 and 7 show the error norm of the eigenvector derivatives corresponding to the repeated eigenvalues according to the shifts  $\Delta\lambda_1/\lambda_1$  and  $\Delta\lambda_2/\lambda_2$  after one iteration, and Figures 6(a)~(d) and figures 8(a)~(d) the ratio according to the number of iterations for various shifts. The number of iterations according to the shift which is required to satisfy the predetermined error norm  $\epsilon_0$  is given in Table 9 and 10. The starting matrix  $Z^{(0)}$  is arbitrary, thus a zero matrix was used in this example. For example, if  $\Delta\lambda_j/\lambda_j$  ( $j=1, 2$ ) is smaller than  $10^{-6}$ , the error norms of the first and second eigenvector derivatives are smaller than  $1.36 \times 10^{-9}$  and  $2.06 \times 10^{-10}$ , respectively, and the ratios for the first and second eigenvector derivatives are almost 1.0. If a shift is taken as  $10^{-16} < \Delta\lambda_j/\lambda_j < 10^{-6}$  ( $j=1, 2$ ),

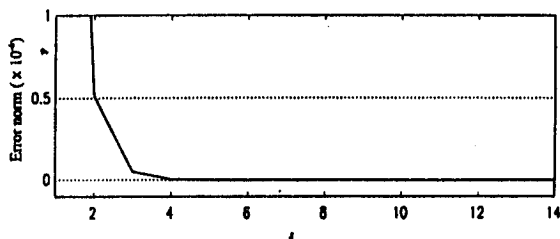


Figure 5. Error norm of the first eigenvector derivative according to the shift after one iteration. Shift  $\Delta\lambda_1/\lambda_1 = 10^{-5}$ .

almost exact results can also be obtained after only one iteration.

The efficiency of the proposed method has been demonstrated in these examples. It is evident that an almost exact solutions would be obtained by the proposed method if a computer having a larger number of bits per numeral were used.

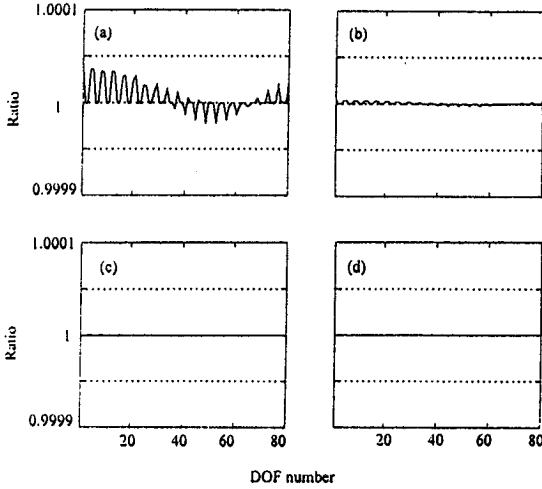


Figure 6. Ratio of each element of the first approximate and exact eigenvector derivative after one iteration. ( $\lambda_1 = 42.9823$ ). (a) When the shift  $\Delta\lambda_1 / \lambda_1 = 10^{-1}$ ; (b)  $10^{-2}$ ; (c)  $10^{-3}$  (d)  $10^{-4}$ .

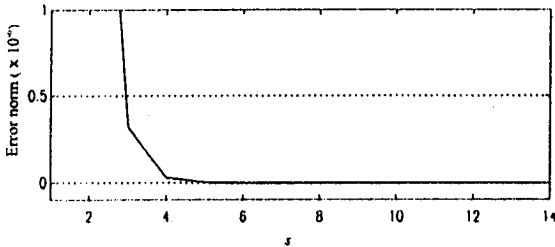


Figure 7. Error norm of the second eigenvector derivative according to the shift after one iteration. Shift  $\Delta\lambda_1 / \lambda_1 = 10^{-s}$ .

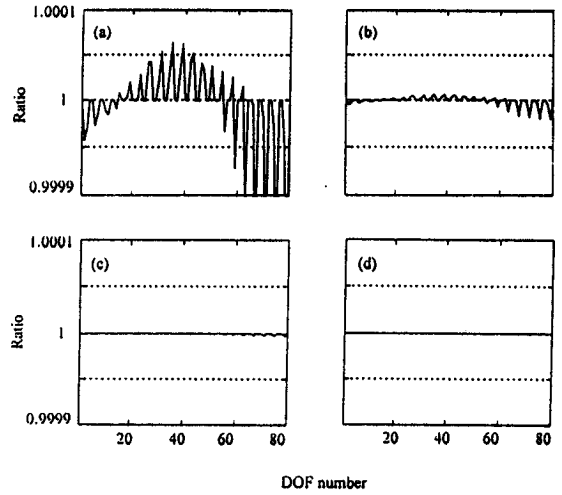


Figure 8. Ratio of each element of the second approximate and exact eigenvector derivative after one iteration. ( $\lambda_2 = 42.9823$ ) (a) When the shift  $\Delta\lambda_2 / \lambda_2 = 10^{-1}$  (b)  $10^{-2}$   $10^{-3}$  (d)  $10^{-4}$ .

Table 9. The number of iterations required to satisfy  $\epsilon_0$  for the first eigenvector derivative of the cantilever beam.

Tolerance $\epsilon_0$	Shift ( $\Delta\lambda_1 / \lambda_1$ ), $\lambda_1 = 42.9823$												
	$10^{-1}$	$10^{-2}$	$10^{-3}$	$10^{-4}$	$10^{-5}$	$10^{-6}$	$10^{-7}$	$10^{-8}$	$10^{-9}$	$10^{-10}$	$10^{-11}$	$10^{-13}$	$10^{-14}$
$10^{-6}$	2	1	1	1	1	1	1	1	1	1	1	1	1
$10^{-9}$	3	2	2	2	1	1	1	1	1	1	1	1	1
$10^{-12}$	3	2	2	2	2	2	2	2	1	1	1	1	1

Table 10. The number of iterations required to satisfy  $\epsilon_0$  for the second eigenvector derivative of the cantilever beam.

Tolerance $\epsilon_0$	Shift ( $\Delta\lambda_2 / \lambda_2$ ), $\lambda_2 = 42.9823$												
	$10^{-1}$	$10^{-2}$	$10^{-3}$	$10^{-4}$	$10^{-5}$	$10^{-6}$	$10^{-7}$	$10^{-8}$	$10^{-9}$	$10^{-10}$	$10^{-11}$	$10^{-13}$	$10^{-14}$
$10^{-6}$	2	2	1	1	1	1	1	1	1	1	1	1	1
$10^{-9}$	3	2	2	2	2	1	1	1	1	1	1	1	1
$10^{-12}$	3	3	2	2	2	2	2	2	1	1	1	1	1

## 5. CONCLUSIONS

An iterative scheme with guaranteed convergence for calculating vibration mode shape derivatives with respect to the design

parameter has been proposed. The proposed method, as well as Nelson's method, has the desirable properties of preserving the symmetries of the stiffness and mass matrices, and of requiring knowledge of only one eigenpair. The proposed iteration scheme can be applied to both non-repeated and repeated eigenvalue problems. When the shift is selected properly, it has been shown through the numerical examples that satisfactory results can be obtained after only one iteration. The complicated procedure for determining the coefficient matrix  $C$  in the case of repeated eigenvalues and the clumsy procedure for finding the particular solution  $V$  encountered in both Nelson's method and Dailey's method are not needed in the proposed iteration scheme. The proposed method can easily find the eigenvector derivatives with repeated eigenvalues without knowledge of the second derivatives of the stiffness and mass matrices. If the algorithm of the proposed method is used on a computer having a larger number of bits per numeral, it is obvious that almost exact results can be obtained because the computer accuracy is higher.

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