중복 고유치를 갖는 비비례 감쇠계의 고유치와 고유벡터의 민감도 해석법

Natural Frequency and Mode Shape Sensitivities of Non-Proportionally Damped Systems: Part II, Multiple Natural Frequencies

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요 지

본 연구에서는 중복 고유치를 갖는 비비례 감쇠 진동계의 고유치와 고유벡터의 민감도를 계산하는 새로운 방법을 제시하였다. 제안 방법은 매우 간단하면서도 수치적 안정성이 보장되고 정확한 해를 주는 방법이다. 제안 방법에서는 $(n+m)$차의 대칭 행렬로 이루어진 대수방정식을 해석함으로써 n개의 자유도를 갖는 감쇠계에 있어서 m차의 중복도를 갖는 고유치와 고유벡터의 설계변수에 대한 미분을 구한다. 제안 방법의 검증을 위해 5자유도를 갖는 단순구조물의 민감도해석을 예제에서 다루고 있다. 예제에서의 설계변수는 모델의 부분강성으로 하였다.

핵심용어 : 중복 고유치, 고유 벡터, 민감도, 설계변수, 부가조건

Abstract

An efficient algorithm whose numerical stability is proved is derived for computation of eigenvector derivatives of non-proportionally damped vibratory systems with multiple eigenvalues. In the proposed method, adjacent eigenvectors and orthonormal conditions are used to compose an algebraic equation of order $n+m$, where n is the number of coordinates and m number of multiplicity. The mode shape derivatives of the damped systems can be obtained by solving the algebraic equation. As an example of a structural system to demonstrate the theory of the proposed method and its possibilities in the case of multiple eigenvalues, the 5-DOF mechanical system is considered. The design parameter of the system is a spring.

Keywords : *multiple eigenvalue, eigenvector, sensitivity, design parameter, side condition*

1. INTRODUCTION

The eigenpair sensitivities of a structural and mechanical systems with multiple natural frequencies have been a focus of recent interest. In typical structural or mechanical syste-

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ms, there are many multiple or nearly equal natural frequencies, due to their structural symmetries or certain reasons. In this case, since eigenspace spanned by the mode shapes corresponding to the multiple natural frequencies is degenerate, any linear combination of mode shapes can be a mode shape. Since the multiple natural frequencies cause the decrease of the rank of the matrix equation used in sensitivity analysis, it is more difficult to get the sensitivities of eigenpairs of a system with multiple natural frequencies than the case of distinct natural frequencies. A number of papers[1)~8)] were presented to find the mode shape derivatives in the case of multiple natural frequencies. To find the derivatives of mode shapes, the adjacent mode shapes must be first calculated which lie "adjacent" to the m (multiplicity of multiple natural frequency) distinct mode shapes appearing when a design parameter varies. To do so, the approximate mode shapes could be varied continuously with varying the design parameter. For the real symmetric case, a generalization of Nelson's method[9)] was obtained by Ojalvo[1)] and amended by Mills-Curren[2)] and Dailey[3)]. Dailey's method is an exact analytical method for calculating the derivatives of mode shapes with multiple eigenvalues. This method only requires the knowledge of eigenpairs to be differentiated, however, the method is lengthy and complicate. Dailey's method is extremely complicate for calculating the sensitivity of eigenvectors of multiple eigenvalues in the case of the damped systems.

In this paper the algebraic method for calculating the derivatives of mode shapes worked by Lee and Jung[7),8)] is extended to the non-proportionally damped systems with multiple natural frequencies. In the case of multiple eigenvalues as well as distinct ones, the proposed method can find the mode shape derivatives by solving an algebraic equation with symmetric coefficient matrix added with side conditions. The orthonormal condition and a set of adjacent eigenvectors can be used in the algebraic equation as side conditions.

The second section of this paper presents the proposed sensitivity analysis method of damped systems with multiple natural frequencies. The third section presents numerical stability of the proposed method, and the next section numerical examples.

2. SENSITIVITY ANALYSIS OF DAMPED SYSTEM WITH MULTIPLE NATURAL FREQUENCIES

When a natural frequency has multiplicity m and a design parameter is perturbed, the corresponding mode shapes may split into as many as m distinct mode shapes. For derivatives of the mode shapes to be responsible, the mode shapes must be lain adjacent to the m distinct mode shapes that appear when a design parameter varies. Otherwise, the mode shapes would jump discontinuously with varying design parameter. Therefor, the first step in finding derivatives of mode shapes of multiple eigenvalues is to find corresponding adjacent mode shapes.

The eigenvalue problem associated with a damped system, in the case of multiple eigenvalues, can be expressed as

$$\mathbf{M}\boldsymbol{\Phi}_m\boldsymbol{\Lambda}_m^2 + \mathbf{C}\boldsymbol{\Phi}_m\boldsymbol{\Lambda}_m + \mathbf{K}\boldsymbol{\Phi}_m = 0 \qquad (1)$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the matrices of mass, damping and stiffness respectively, and these are order n symmetric matrices. $\mathbf{M}$ is positive definite and $\mathbf{K}$ positive definite or semi-positive definite. $\mathbf{\Lambda}_m$ is a eigenvalue matrix having eigenvalues, λ_m's, of multiplicity m on its diagonal, and $\mathbf{\Phi}_m$ is the matrix of eigenvectors corresponded to the multiple eigenvalue hence its order $(n \times m)$. As referred in Part I, the orthonormal condition for the matrix $\mathbf{\Phi}_m$ is as follows :

$$\mathbf{\Phi}_m^T(2\lambda_m\mathbf{M}+\mathbf{C})\mathbf{\Phi}_m = \mathbf{I}_m \tag{2}$$

The eigenvector matrix $\mathbf{\Phi}_m$ can not be used in calculating the derivatives of eigenvectors since it is not a set of adjacent eigenvectors, generally. The adjacent eigenvectors can be expressed in terms of $\mathbf{\Phi}_m$ by an orthogonal transformation such as

$$\mathbf{X} = \mathbf{\Phi}_m\mathbf{T} \tag{3}$$

where $\mathbf{T}$ is an orthonormal transformation matrix, $\mathbf{T}^T\mathbf{T} = \mathbf{I}_m$, and its order m. The columns of $\mathbf{X}$ are the adjacent eigenvectors. It is natural that the adjacent eigenvectors satisfy the orthonormal condition too ;

$$\begin{aligned}\mathbf{X}^T(2\lambda_m\mathbf{M}+\mathbf{C})\mathbf{X} &= \mathbf{T}^T\mathbf{\Phi}_m^T(2\lambda_m\mathbf{M}+\mathbf{C})\mathbf{\Phi}_m\mathbf{T}\\ &= \mathbf{T}^T\mathbf{T} = \mathbf{I}_m\end{aligned} \tag{4}$$

The next procedure is to find $\mathbf{T}$ and then to find $\mathbf{X}$. Consider the following another eigenvalue problem.

$$\mathbf{M}\mathbf{X}\mathbf{\Lambda}_m^2+\mathbf{C}\mathbf{X}\mathbf{\Lambda}_m+\mathbf{K}\mathbf{X}=0 \tag{5}$$

Differentiating the above eigenvalue problem with respect to the design parameter p, and rearranging yield

$$\begin{aligned}[\lambda_m^2\mathbf{M}+\lambda_m\mathbf{C}+\mathbf{K}]\frac{\partial \mathbf{X}}{\partial p} &= -(2\lambda_m\mathbf{M}+\mathbf{C})\frac{\partial \mathbf{\Lambda}_m}{\partial p}\\ &\quad -\left(\lambda_m^2\frac{\partial \mathbf{M}}{\partial p}+\lambda_m\frac{\partial \mathbf{C}}{\partial p}+\frac{\partial \mathbf{K}}{\partial p}\right)\mathbf{X}\end{aligned} \tag{6}$$

Premultiplying at each side of equation (6) by $\mathbf{\Phi}_m^T$ and substituting $\mathbf{X}=\mathbf{\Phi}_m\mathbf{T}$ into it give new eigenvalue problem such as

$$\mathbf{D}\mathbf{T} = \mathbf{E}\mathbf{T}\frac{\partial \mathbf{\Lambda}_m}{\partial p} \tag{7}$$

where

$$\mathbf{D}=\mathbf{\Phi}_m^T\left(\lambda_m^2\frac{\partial \mathbf{M}}{\partial p}+\lambda_m\frac{\partial \mathbf{C}}{\partial p}+\frac{\partial \mathbf{K}}{\partial p}\right)\mathbf{\Phi}_m \quad \text{and}$$

$$\mathbf{E} = -\mathbf{\Phi}_m^T(2\lambda_m\mathbf{M}+\mathbf{C})\mathbf{\Phi}_m = -\mathbf{I}_m \tag{8}$$

One can obtain the eigenvalue derivative $\partial\mathbf{\Lambda}_m/\partial p$ and orthogonal transformation matrix $\mathbf{T}$ by solving equation (7), and then the adjacent eigenvectors by relation $\mathbf{X}=\mathbf{\Phi}_m\mathbf{T}$.

The proposed method starts with equation (6) and equation (4). Differentiating the orthonormal condition, equation (4), with respect to design parameter gives

$$\begin{aligned}\mathbf{X}^T(2\lambda_m\mathbf{M}+\mathbf{C})\frac{\partial \mathbf{X}}{\partial p} &= -\mathbf{X}^T\frac{\partial \mathbf{X}}{\partial p}\mathbf{X}\mathbf{\Lambda}_m\\ &\quad -\mathbf{X}^T\mathbf{M}\mathbf{X}\frac{\partial \mathbf{\Lambda}_m}{\partial p}-\frac{1}{2}\mathbf{X}^T\frac{\partial \mathbf{C}}{\partial p}\mathbf{X}\end{aligned} \tag{9}$$

One can write the following single matrix equation by combining equation (6) and equation (9).

$$\begin{bmatrix}\lambda_m^2\mathbf{M}+\lambda_m\mathbf{C}+\mathbf{K} & (2\lambda_m\mathbf{M}+\mathbf{C})+\mathbf{X}\\ \mathbf{X}^T(2\lambda_m\mathbf{M}+\mathbf{C}) & \mathbf{0}\end{bmatrix}\begin{bmatrix}\frac{\partial \mathbf{X}}{\partial p}\\ \mathbf{0}\end{bmatrix}$$

$$\begin{bmatrix}-(2\lambda_m\mathbf{M}+\mathbf{C})\mathbf{X}\frac{\partial \mathbf{\Lambda}_m}{\partial p}-\left(\lambda_m^2\frac{\partial \mathbf{M}}{\partial p}+\lambda_m\frac{\partial \mathbf{C}}{\partial p}+\frac{\partial \mathbf{K}}{\partial p}\right)\mathbf{X}\\ -\mathbf{X}^T\frac{\partial \mathbf{M}}{\partial p}\mathbf{X}\mathbf{\Lambda}_m-\mathbf{X}^T\mathbf{M}\mathbf{X}\frac{\partial \mathbf{\Lambda}_m}{\partial p}-\frac{1}{2}\mathbf{X}^T\frac{\partial \mathbf{C}}{\partial p}\mathbf{X}\end{bmatrix} \tag{10}$$

The derivatives of the adjacent eigenvectors, $\partial \mathbf{X}/\partial p$, can be found by solving equation (10). The coefficient matrix in the left-hand side of equation (10) is always nonsingular (see the next section)

Note that the proposed method has the desirable properties of preserving the structure of the system matrices, and of requiring knowledge of only multiple eigenpairs. Note also that the proposed method needs the first order derivatives of the mass, damping and stiffness matrices, whereas Dailey's method needs the first and second derivatives of them.

3. NUMERICAL STABILITY OF THE PROPOSED METHOD

Identifying the nonsingularity of the coefficient matrix $\mathbf{A}^*$ in equations (10) and (11) may prove the numerical stability of the proposed method in the case of multiple eigenvalues.

$$\mathbf{A}^* = \begin{bmatrix} \lambda_m^2\mathbf{M}+\lambda_m\mathbf{C}+\mathbf{K} & (2\lambda_m\mathbf{M}+\mathbf{C})\mathbf{X} \\ \mathbf{X}^T(2\lambda_m\mathbf{M}+\mathbf{C}) & \mathbf{0} \end{bmatrix} \tag{11}$$

To show that the coefficient matrix $\mathbf{A}^*$ is always nonsingular, consider another matrix such as $\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}$ where $\mathbf{Y}$ is a $(n+m)\times(n+m)$ nonsingular matrix. In this paper, the matrix $\mathbf{Y}$ is assumed as

$$\mathbf{Y} = \begin{bmatrix} \mathbf{\Psi} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \tag{12}$$

where $\mathbf{I}_m$ is an identity matrix of order m and $\mathbf{\Psi}$ is a set of arbitrary independent vectors containing the adjacent eigenvectors of multiple eigenvalue λ_m of the system, as follows

$$\mathbf{\Psi} = [\psi_1\ \psi_2\ \cdots\ \psi_{n-m}\ \mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_m] \quad \text{when}$$

$$\mathbf{X} = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_m] \tag{13}$$

where ψ's are arbitrary independent vectors chosen to be independent to the adjacent eigenvector x's. Pre- and post-multiplying $\mathbf{Y}^T$ and $\mathbf{Y}$ to $\mathbf{A}^*$ yield

$$\mathbf{Y}^T\mathbf{A}^*\mathbf{Y} = \begin{bmatrix} \mathbf{\Psi}^T(\lambda_m^2\mathbf{M}+\lambda_m\mathbf{C}+\mathbf{M})\mathbf{\Psi} & \mathbf{\Psi}^T(2\lambda_m\mathbf{M}+\mathbf{C})\mathbf{X} \\ \mathbf{X}^T(2\lambda_m\mathbf{M}+\mathbf{C})\mathbf{\Psi} & \mathbf{0} \end{bmatrix} \tag{14}$$

Considering the eigenvalue problem $(\lambda_m^2\mathbf{M}+\lambda_m\mathbf{C}+\mathbf{K})\mathbf{X}=\mathbf{0}$ yields

$$\mathbf{\Psi}^T(\lambda_m^2\mathbf{M}+\lambda_m\mathbf{C}+\mathbf{K})\mathbf{\Psi} = \begin{bmatrix} \tilde{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{15}$$

where $\tilde{\mathbf{A}}$ is non-zero $(n-m)\times(n-m)$ submatrix. The submatrix $\tilde{\mathbf{A}}$ is a nonsingular matrix, $\det(\tilde{\mathbf{A}})\neq 0$, having order of $n-m$ and rank of $n-m$, since it is given by eliminating the columns and rows having all zero elements from $\mathbf{\Psi}^T(\lambda_m^2\mathbf{M}+\lambda_m\mathbf{C}+\mathbf{K})\mathbf{\Psi}$ of order n and rank $n-m$. The orthonormal condition yields

$$\mathbf{\Psi}^T(2\lambda_m\mathbf{M}+\mathbf{C})\mathbf{X} = \begin{Bmatrix} \tilde{\mathbf{B}} \\ \mathbf{I}_m \end{Bmatrix} \quad \text{and}$$

$$\mathbf{X}^T(2\lambda_m\mathbf{M}+\mathbf{C})\mathbf{\Psi} = \begin{Bmatrix} \tilde{\mathbf{B}} \\ \mathbf{I}_m \end{Bmatrix}^T \tag{16}$$

where $\tilde{\mathbf{B}}$ is generally non-zero rectangular matrix. Substituting equations (15) and (16), into equation (14) yields

$$\mathbf{Y}^T\mathbf{A}^*\mathbf{Y} = \begin{bmatrix} \tilde{\mathbf{A}} & \mathbf{0} & \tilde{\mathbf{B}} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_m \\ \tilde{\mathbf{B}}^T & \mathbf{I}_m & \mathbf{0} \end{bmatrix} \tag{17}$$

By applying the matrix determinant property of partitioned matrices, the determinant of can be rewritten as

$$\det(\mathbf{Y}^T\mathbf{A}^*\mathbf{Y})=\det\begin{bmatrix}0 & \mathbf{I}_m\\ \mathbf{I}_m & 0\end{bmatrix}$$

$$\det\left(\tilde{\mathbf{A}}-[0\ \ \tilde{\mathbf{B}}]\begin{bmatrix}0 & \mathbf{I}_m\\ \mathbf{I}_m & 0\end{bmatrix}^{-1}\begin{bmatrix}0\\ \tilde{\mathbf{B}}^T\end{bmatrix}\right) \qquad (18)$$

or

$$\det(\mathbf{Y}^T\mathbf{A}^*\mathbf{Y})=\det(\tilde{\mathbf{A}})\neq 0 \qquad (19)$$

The determinant of $\mathbf{A}^*$ thus is not equal to zero because $\det(\mathbf{Y}^T\mathbf{A}^*\mathbf{Y})\neq 0$. The proof is completed mathematically for the numerical stability of the proposed algorithm in the case of multiple eigenvalues.

4. NUMERICAL EXAMPLES

An analytical example to verify the proposed method, primary and secondary systems equipped on the rigid square plate with the 5-DOF mass, spring and damper showed in Fig. 1 is considered. Assume that only the vertical vibrations are allowed. As shown in the following analysis results, the initial structure has multiple eigenvalues due to its structural symmetry. The lumped dampers cause the non-proportional damping matrix in the equation of motion.

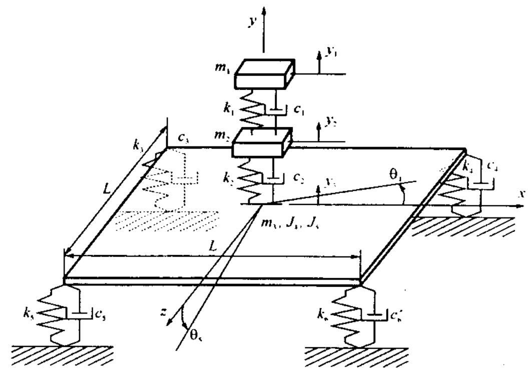

Fig. 1 5-DOF non-proportionally damped system.

$m_1=200kg,\ m_2=500kg,\ m_3=1000kg$;
$k_1=10000\ N/m,\ k_2=20000N/m,$
$k_3=k_4=k_5=k_6=1000\ N/m$; $c_1=4Ns/m,$
$c_2=6Ns/m,\ c_3=c_4=c_5=c_6=40Ns/m$;

Some results are summarized in Table 1.

Table 1 The natural frequencies of the initial and changed primary and secondary system, and results of the sensitivity analysis.

Mode Number	Initial System		Changed System		Variation of Eigenpair		Error of Approximation	
	Eigenvalue	Eigenvalue Derivative	Eigenvalue	Approximated Eigenvalue	Eigenvalue	Eigenvector	Eigenvalue	Eigenvector
1, 2	-4.3262×10^{-2} $\pm j1.5023\times10^{0}$	9.6943×10^{-7} $\pm j1.7995\times10^{-4}$	-4.3253×10^{-2} $\pm j1.5040\times10^{0}$	-4.3253×10^{-2} $\pm j1.5041\times10^{0}$	1.1893×10^{-3}	4.6721×10^{-3}	8.1631×10^{-6}	2.9463×10^{-5}
3, 4	-2.4000×10^{-1} $\pm j3.4558\times10^{0}$	0.0000×10^{0} $\pm j0.0000\times10^{0}$	-2.4000×10^{-1} $\pm j3.4558\times10^{0}$	-2.4000×10^{-1} $\pm j3.4558\times10^{0}$	0.0000×10^{0}	0.0000×10^{0}	0.0000×10^{0}	0.0000×10^{0}
5, 6	-2.4000×10^{-1} $\pm j3.4558\times10^{0}$	0.0000×10^{0} $\pm j8.6811\times10^{-4}$	-2.4000×10^{-1} $\pm j3.4645\times10^{0}$	-2.4000×10^{-1} $\pm j3.4645\times10^{0}$	2.5039×10^{-3}	1.5461×10^{-3}	2.1632×10^{-6}	5.2014×10^{-6}
7, 8	-3.5202×10^{-2} $\pm j6.1354\times10^{0}$	-7.8926×10^{-7} $\pm j2.9526\times10^{-5}$	-3.5210×10^{-2} $\pm j6.1357\times10^{0}$	-3.5210×10^{-2} $\pm j6.1357\times10^{0}$	4.8257×10^{-5}	1.0987×10^{-3}	1.1763×10^{-7}	2.5394×10^{-6}
9, 10	-2.4535×10^{-2} $\pm j9.7000\times10^{0}$	-1.8017×10^{-7} $\pm j5.0001\times10^{-6}$	-2.4537×10^{-2} $\pm j9.7000\times10^{0}$	-2.4537×10^{-2} $\pm j9.7000\times10^{0}$	5.1624×10^{-6}	1.9422×10^{-4}	4.3893×10^{-9}	1.6332×10^{-7}

Table 1 shows the eigenvalues of the system and their sensitivities when design parameter is k_5. Note that the second and third eigenvalue conjugate pairs are multiple respectively. The derivatives of the multiple eigenvalues are different since the design parameter is the spring k_5; when k_5 is varied, the multiple eigenvalues are split into distinct ones since the structural symmetry is broken. It is natural that the second eigenvalue conjugate pair has zero derivatives, since corresponding eigenvectors have zero displacements on k_5 and so the variation in k_5 makes no effects on the modes. The exact and approximated eigenvalues of the system after changing k_5 by $\Delta k_5/k_5=0.01$ are represented in the fourth and fifth columns of the table. The last four columns are variations of exact eigenpairs and errors of the approximate eigenpairs. Since the sensitivities of the second and third eigenpairs are equal to zero, the second and third eigenpairs are not changed. Considering that the errors of the approximate eigenpairs are relatively smaller than the variation of k_5, the approximate eigenvalues and eigenvectors of the changed system are reasonable. Consequently, one can say that the proposed method gives good results.

5. CONCLUSIONS

This paper proposes an efficient numerical method for calculating vibration mode shape derivatives of the non-proportionally damped systems with multiple eigenvalues. The method finds eigenpair derivatives of the systems by solving the linear algebraic equation without any numerical instability. The proposed method is very efficient in the case of the multiple eigenvalue problems since the computer storage and analysis time are saved in comparison with Dailey's method, since our method does not use second derivatives of the system matrices while Dailey's method does. The proposed method is simpler than any other methods and gives exact solutions. The proposed method may be inserted easily into the commercial FEM code since it finds the exact solution and treats symmetric matrix. Furthermore, its algorithm is very simple and its numerical stability proved.

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