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A Study on Mode Localization of Non-periodic Structures

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Abstract

The mode localization phenomena in simply supported two span beams of arbitrary span lengths are theoretically investigated. When localization occurs, the free vibration amplitude of a normal mode becomes confined to a local region of the structure, with serious implication for the control problem. In many structures, some modes that are not localized become localized seriously by small structural changes. It is well known that the weakly coupled periodic structures are sensitive to certain types of periodicity-breaking disorder, resulting in the mode localization. In the previous researches, perturbation methods are used to discuss the phenomenon and periodic structures are mainly considered. In this study, however, the mode localization phenomenon is discussed with an analytical approach and it is shown that the mode localization can occur in non-periodic structures also by small structural changes. The results of this study show that the coupling strength plays the important role not only in the mode localization of the periodic structures but also in that of the non-periodic ones. Degrees of mode localization in the two span beams and their sensitivities to system parameters are appraised by considering the characteristic graph and the structural line defined in this study.

Keywords : *mode localization, two-span beam, periodic structure, non-periodic structure, eigenvalue, eigenvector.*

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1. Introduction

The natural frequencies and the mode shapes are the dynamic characteristics of the structural systems, which are functions of the geometric configuration and the material properties of the structures. The dynamic characteristics may be changed extremely by structural changes in some structures when mode localization occurs. The mode localization is a phenomenon that the magnitude of the specific part of the free vibrational mode is large relative to the rest of the mode. It is well known that the weakly coupled periodic structures are sensitive to certain types of periodicity-breaking disorder, resulting in the mode localization with serious implication for the control problem. In the field of mode localization, periodically stiffened long plates, truss structures, multi-span rahmen bridges, bladed rotors of a turbomachinery, etc. are well known periodic structures sensitive to mode localization. For the structures the structural damages or the manufacturing errors may produce the undesirable mode localization. It is, therefore, very important not only to calculate the natural frequencies and the mode shapes but to identify the degrees of localization and the localization sensitivities of the modes.

In solid-state physics, the localization phenomenon of electron field in disordered solid was first observed by Anderson.(1) Anderson and Mott(2) shared the 1977 Nobel Prize in physics for their work in this area. The mode localization phenomenon is found to exist in the field of structural dynamics. Many works were concerned on cyclically symmetric structures with weak coupling in order to explain the unpredicted fatigue failure of the mistuned blades of turbomachinery.(3)-(5) Bendiksen(6) investigated the mode localization in a simple model of a space structure. Hodges(7) was the first to recognize that the wave localization may occur in the disordered periodic structures and it leads to mode localization. The wave localization is the phenomenon that the vibrational energy imparted to the structure by an external source cannot propagate to arbitrary long distances but is instead substantially confined to a region close to the source. After his work, there have been several studies on the localization in periodic engineering structurel(8)-(11). Pierre, Tang and Dowell(9) studied the mode localization of weakly coupled disordered multi-span beams using the modified perturbation method and the experimental method. Bouzit and Pierre(10) demonstrated weak and strong localization behaviors and calculated the localization factor for a multi-span beam on randomly spaced simple supports, the torsional rigidity of which could be varied. The localization factor is defined by the average exponential rate at which a structural wave decays with respect to the wave propagation distance in a disordered periodic structure and it can be calculated by the transformation matrix method. The mode localization of non-linear systems was studied by Vakakis et al,(12),(13) and Zevin.(14) Studies on the mode localization in real structures such as truss and cable systems were presented by Hawwa et al,(15) and Poovarodom and Yamaguchi.(16)

In the previous works for mode localization, the disordered periodic structures are mainly concerned. However, the mode localization phenomenon in the non-periodic structures has been passed over. The first objective of this study is to show the possibility of drastic occurrences of mode localization in non-periodic structures. And the second is to introduce the mode localization phenomena into the field of structural dynamics of civil engineering. Free vibration analysis of simply supported two-span beams of arbitrary span lengths is theoretically investigated. The beam can be periodic or non-periodic. Degrees of mode localization and their sensitivities to system parameters are appraised by considering the characteristic graph and the structural line defined in this study.

2. Free Vibration Analysis of the Two-span beam

Consider the two-span beam shown in Figure 1, which is a simple model of rib-stiffened plates or rahmen bridges. The beam is simply supported at both ends, and is constrained to have zero deflection at $x_1=l_1$ and/or $x_2=0$. Moreover, a torsional spring of K_R exerts a restoring moment at $x_1=l_1$ and/or $x_2=0$. The system can be divide into two substructures and for the convenience of the simple analysis the coordinates of the substructures are determined as in Figure 1.

The eigenvalue problems for free bending vibrations of each substructure can be written as

$$EI_1\frac{d^4y_1}{dx_1^4}-\omega^2m_1y_1=0 \tag{1}$$

and

$$EI_2\frac{d^4y_2}{dx_2^4}-\omega^2m_2y_2=0 \tag{2}$$

where EI_1 and EI_2 are flexural rigidity, m_1 and m_2 are masses per unit length of each substructure respectively, ω is natural frequency of the system, y_1 and y_2 and are the transverse displacements of

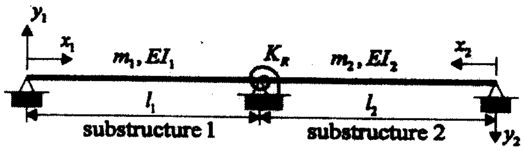


Fig. 1. Simply Supported Two-Span Beam with Rotational Atiffness at Mid Support.

each substructure. The general solutions of Eqs. (1) and (2) can be written as

$$y_1(x_1)=A_1\sin\lambda_1x_1+B_1\cos\lambda x_1+C_1\sinh\lambda x_1 + D_1\cosh\lambda x_1 \tag{3}$$

and

$$y_2(x_2)=A_2\sin\lambda_2x_2+B_2\cos\lambda_2x_2+C_2\sinh\lambda_2x_2 + D_2\cosh\lambda_2x_2 \tag{4}$$

where

$$\lambda_1^4=\omega^2\frac{m_1}{EI_1} \text{ and } \lambda_2^4=\omega^2\frac{m_2}{EI_2} \tag{5, 6}$$

To determine the coefficients of the general solutions, one can use eight boundary conditions. By applying boundary conditions that the deflections and the moments at $x_1=0$ and $x_2=0$ are zeros, Eqs. (3) and (4) yield $B_1=D_1=B_2=D_2=0$. Application of boundary conditions that the deflections at $x_1=l_1$ and $x_2=l_2$ are zeros yield

$$C_1=-A_1\frac{\sin\lambda_1l_1}{\sinh\lambda_1l_1} \text{ and } C_2=-A_2\frac{\sin\lambda_2l_2}{\sinh\lambda_2l_2} \tag{7, 8}$$

Eqs. (7) and (8), and the two continuity conditions such as

$$\frac{dy_1(l_1)}{dx_1}=\frac{dy_2(l_2)}{dx_2} \tag{9}$$

and

$$EI_1\frac{d^2y_1(l_1)}{dx_1^2}+EI_2\frac{d^2y_2(l_2)}{dx_2^2}=-K_R\frac{dy_2(l_2)}{dx_2} \tag{10}$$

give two algebraic equations for A_1 and A_2, and that can be written in a matrix form as

$$\begin{bmatrix} \lambda_1\theta_1\sin\lambda_1 l_1 & -\lambda_2\theta_2\sin\lambda_2 l_2 \\ -2EI_1\lambda_1^2\sin\lambda_1 l_1 & -2EI_2\lambda_2^2\sin\lambda_2 l_2+K_R\lambda_2\theta_2\sin\lambda_2 l_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{11}$$

where

$$\theta_1(\lambda_1 l_1)=\frac{\cos\lambda_1 l_1}{\sin\lambda_1 l_1}-\frac{\cosh\lambda_1 l_1}{\sinh\lambda_1 l_1} \tag{12}$$

and

$$\theta_2(\lambda_2 l_2)=\frac{\cos\lambda_2 l_2}{\sin\lambda_2 l_2}-\frac{\cosh\lambda_2 l_2}{\sinh\lambda_2 l_2} \tag{13}$$

Nontrivial solutions of Eq. (11) can be obtained if and only if the determinant of the its coefficient matrix vanishes. This gives an equation for the determination of natural frequencies, ω, which is called the frequency equation or characteristic equation:

$$K_R\theta_1\theta_2-2EI_2\lambda_2\theta_1-2EI_1\lambda_1\theta_2=0 \tag{14}$$

In Eq. (14), the only unknown is natural frequency ω. However, it is convenient to use two variables, β_1 and β_2, for describing the characteristics of the system. Using β_1 and β_2 we can rewrite the characteristic equation as

$$K_R\theta_1\theta_2-2\frac{EI_2}{l_2}\beta_2\theta_1-2\frac{EI_1}{l_1}\beta_1\theta_2=0 \tag{15}$$

where

$$\beta_1=\lambda_1 l_1=\omega^{\frac{1}{2}}l_1\left(\frac{m_1}{EI_1}\right)^{\frac{1}{4}} \tag{16}$$

and

$$\beta_2=\lambda_2 l_2=\omega^{\frac{1}{2}}l_2\left(\frac{m_2}{EI_2}\right)^{\frac{1}{4}} \tag{17}$$

From Eqs. (16) and (17), we get

$$\beta_2=\alpha\beta_1 \tag{18}$$

where

$$\alpha=\frac{l_2}{l_1}\left(\frac{m_2}{EI_2}\,\frac{EI_1}{m_1}\right)^{\frac{1}{4}} \tag{19}$$

Eq. (18) is an equation for line α and is the slope of the line. In this study, the line and the slope were named structural line and structural slop respectively since they represent the geometry and material properties of the structure. Those are very useful to describe the characteristic of the system, and will be used in the next section.

3. Mode Localization of the Two-Span Beam

In this section the mode localization factor and the characteristic graph are defined here. The mode localization factor can be used for measure of the degree of mode localization of each mode. And using the characteristic graph, one can roughly forecast the effects of the changes in the system parameters on the mode localization.

As aforementioned, the mode localization is the vibration energy confinement, so the degree of mode localization can be represented by logarithmic value of the ratio of the mean squared vibrational magnitude of the second substructure to that of the first substructure as

$$\gamma\equiv\log\frac{\eta_2}{\eta_1} \tag{20}$$

where γ is the mode localization factor, η_1 and η_2 and are the mean squared free vibrational magnitude of the first and second substructures respectively, which can be expressed as

$$\eta_1\equiv\frac{1}{l_1}\int_0^{l_1}y_1^2(x_1)dx_1 \tag{21}$$

and

$$\eta_2\equiv\frac{1}{l_2}\int_0^{l_2}y_2^2(x_2)dx_2 \tag{22}$$

For simple analysis, considering $\beta\gg1$, one can conclude that $C_1\approx0$ and $C_2\approx0$ from Eqs. (7) and (8), and get approximated γ as

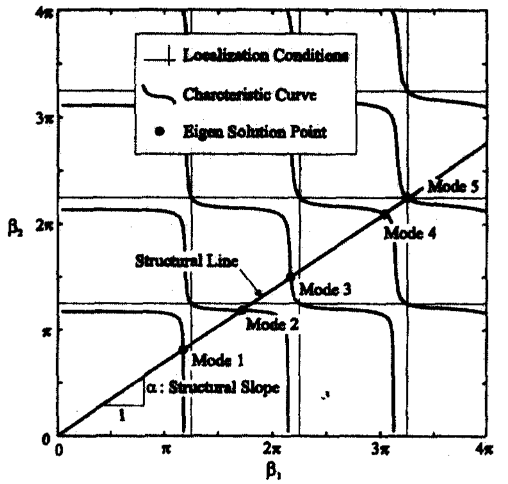


Fig. 2. Characteristic Graph of the Two-Span Beam.

$$\bar{\gamma} = \log \frac{A_2^2}{A_1^2} \tag{23}$$

If the vibration is confined at one of the substructures, the absolute value of γ becomes large. On the contrary, if the vibrational magnitudes are the same with each other, $|A_1|=|A_2|$, $\bar{\gamma}$ becomes zero. The sign of $\bar{\gamma}$ means the substructure at which the vibration is confined, positive at second substructure and negative at first one. Considering Eq. (11) and an assumption that two substructures have the same material properties, one can rewrite Eq. (23) as

$$\bar{\gamma}=\log\left(\cos\beta_1-\frac{\cosh\beta_1}{\sinh\beta_1}\sin\beta_1\right)^2 -\log\left(\cos\beta_2-\frac{\cosh\beta_2}{\sinh\beta_2}\sin\beta_2\right)^2 \tag{24}$$

The mode localization factor γ is close to zero when $(\beta_1-\beta_2)=n\pi$ and positive or negative infinite when $(\beta_1-\beta_2)=(n+0.5)\pi$, where n is zero or integer. If only one of the following conditions is satisfied, the absolute value of $\bar{\gamma}$ is to be large and the mode is localized.

$$\beta_1\approx\left(m+\frac{1}{4}\right)\pi \tag{25}$$

and

$$\beta_2\approx\left(m+\frac{1}{4}\right)\pi \tag{26}$$

where

$m=1, 2, 3, \dots$

A mode was considered as localized one when the span response ratio is less than 0.1 in Ref 9. To be consistent with that terminology, a mode is considered as localized one when $|\bar{\gamma}|\geq 2$.

Considering the above localization conditions and the results of the previous section, one can draw a characteristic graph as in Fig. 2.

In Fig. 2, the horizontal and the vertical axes represent β_1 and β_2 respectively. The characteristic curves represent the characteristic equation, Eq. (15), and the structural line and the structural slope represent Eqs. (18) and (19) respectively. The localization conditions that are the vertical and horizontal thin solid lines mean Eqs. (25) and (26) respectively. The crossing points of the characteristic curves and the structural line indicate the eigen solutions of the system. By using β_1 or β_2 of the eigen solution points and Eqs.(16) or (17), one can calculate the natural frequencies of the system. The more an eigen solution is close to one of the thin solid lines and is far from the other ones, the more the mode is strongly localized. However, if an eigen solution is close to the vertical and the horizontal localization condition lines concurrently or on the crossing point of them, the mode is not to be localized since the points satisfy $(\beta_1-\beta_2)=n\pi$.

For examples, in Fig. 2, the first, second and third modes are localized while the fourth and fifth modes are not. Because first three modes are close to one of the localization conditions, but the fourth mode is close to two localization conditions and the fifth one is on them at a time. Especially the first and the third mode are close to the vertical localization conditions. That means that β_1 's of them satisfy Eq. (25), so it is sure that the mode localization factor of them have negative large values and the vibration is to be confined at the first substructure of the system. And for the second mode, it is close to the horizontal localization condition, so the vibration is to be localized at the second substructure.

What makes the best use of the characteristic graph is that one can roughly predict the mode localization phenomenon occurred by any disturbances introduced into a two-span beam. Small changes in system parameters produce the changes in the structural slope and the characteristic curves, but the variation in characteristic curves are small. The change in structural slope shifts the eigen solution points and the degrees of mode localization of each mode are varied also. Since the structural line start at the origin, the shifting caused by the change in slope is more and more steep with the mode number. For such reason, the shifting of higher modes can be very serious and the variation in degree of mode localization also. Considering the system depicted in Fig. 2, for instance, one can say that if the structural slope is increased by some disturbances, all the eigen solution points are shifted. And as a result of that the degrees of mode localization of the first and the second modes are decreased while the others are increased.

4. Examples

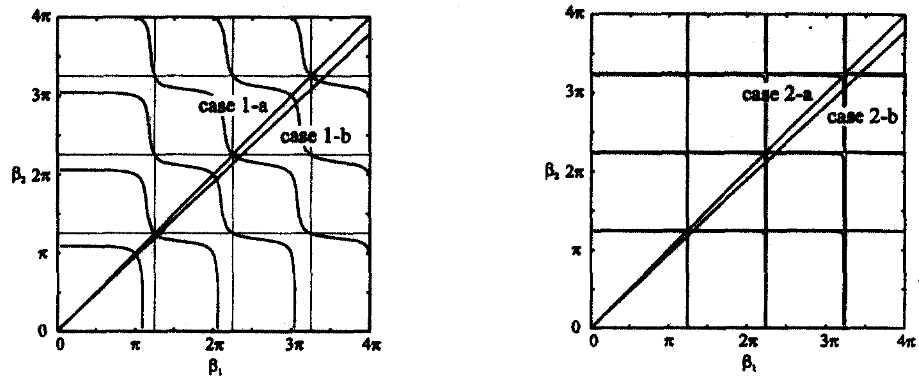
4.1. The System Considered

In this section the foregoing results are confirmed by some examples. In this example, the mode localization phenomenon caused by the small changes in system parameters in the strongly or weakly coupled periodic and non-periodic systems are discussed by using characteristic graphs.

The example structures are continuous beam structures resting on the three simple supports, which are constrained by torsional spring at mid support as shown schematically in Fig. 1. The torsional spring plays the role of a decoupler. As $K_R \to \infty$, the spans are fully decoupled from each other because no moment can be transmitted fromonesubstructuretotheother.For$K_R=0$, the substructures are strongly coupled since no restoring moment is exerted. The effects of the coupling and the periodicity on the mode localization are studied by considering four cases. In the first two cases the effect of the coupling on mode localization in periodic structure is studied, and in the next two cases the non-periodic structure is considered. All the example structures have the same material properties and each of the cases is classified by the strength of the torsional spring and the span length of the second substructure. However one can study the effects of any other material properties using the same procedure presented in this paper, and will get similar results because the non-dimensional parameters β_1 and β_2 are mainly used in the procedure.

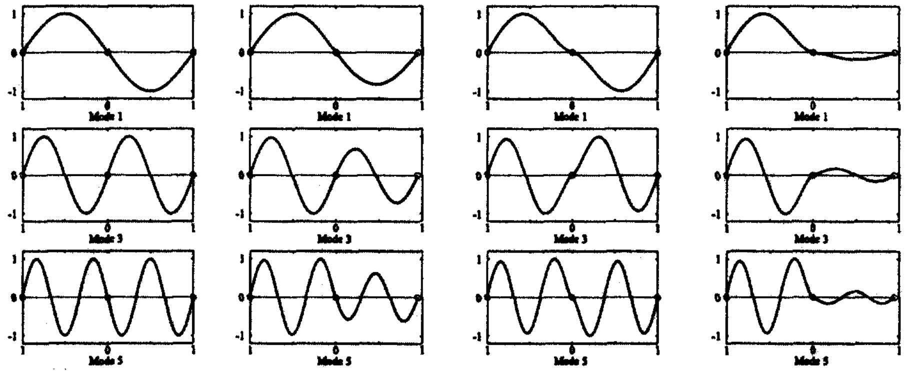
4.2. The Periodic Structures; Cases 1 and 2.

The mode localization phenomena in the ordered and the disordered periodic two-span beams are examined for two coupling conditions. In the disordered cases, the ratio of the span length of the second substructure to that of the first one is 0.95 while the ratio is



(a) case 1−a: α=1.0; case 1−b: α=0.95 (b) case 2−a: α=1.0; case 2−b: α=0.95.

Fig. 3. Characteristic Graph of Case 1 and 2.



(a) Case 1−a (b) Case 1−b (c) Case 2−a (d) Case 2−b

Fig. 4. Mode Shapes of Case 1 and 2.

unity in the ordered cases. The masses per unit length of each substructure are $m_1=m_2=25.0\ kg/m$, the flexural rigidities $EI_1=EI_2=2\times10^7 Nm^2$, and the span length $l_1=l_2=1.0\ m$ in ordered cases while $l_2=0.95\ m$ in disordered ones. The torsional spring constants are $K_R=0.00Nm$ and $K_R=2\times10^8 Nm$ in cases 1 and 2 respectively. The substructures of case 1 are strongly coupled with each other since they have no torsional spring, and the next case are weakly coupled system. The results of the ordered and the disordered cases are discussed in pair because this example performed on the assumption that some errors such as manufacturing errors or structural damages make an ordered system into disordered one. Fig. 3 shows the characteristic graph of the cases 1 and 2. Subcases *a* and *b* imply an ordered system and a disordered one respectively. Selected mode shapes of each case are shown in Fig. 4. The lowest five natural frequencies and the degrees of mode localization of each case and its differences are given in table 1 and 2.

All the eigen solution points of the case 1−a and case 2−a satisfy $(\beta_1-\beta_2)=n\pi$ where $n=0$ as shown in Fig. 3, and it is clear that all the modes are not localized as shown in Fig. 4. However, for the disordered cases such as case 1−b and case 2−b, $(\beta_1-\beta_2)=n\pi$ is not satisfied and all the modes are localized in some degrees as shown in Fig. 4. The eigen so-

Table 1. Natural frequencies and degrees of mode localization : case 1.

Mode	Case 1-a		Case 1-b		Differences	
	$f(Hz)$	$\bar{\gamma}$	$f(Hz)$	$\bar{\gamma}$	$\delta f(\%)$	$\delta\bar{\gamma}$
1	444.28	0.0000	465.46	-0.1384	4.77	-0.1384
2	694.06	0.0000	733.86	0.1731	5.73	0.1731
3	1777.2	0.0000	1854.7	-0.2695	4.36	-0.2695
4	2249.2	0.0000	2387.3	0.3032	6.14	0.3032
5	3998.6	0.0000	4158.4	-0.3880	4.00	0.3880

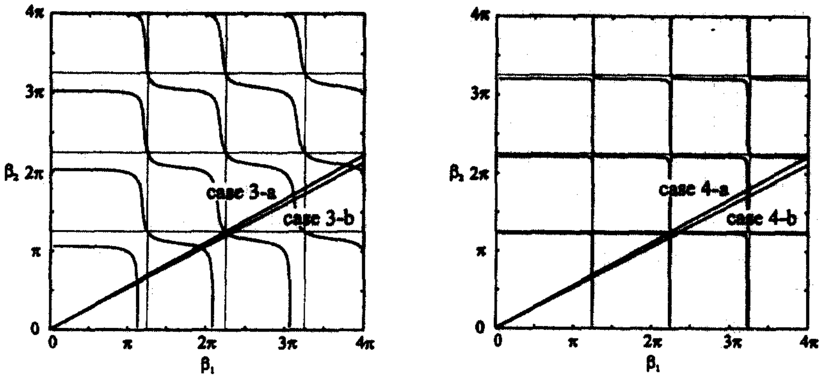
Table 2. Natural frequencies and degrees of mode localization: case 2.

Mode	Case 2-a		Case 2-b		Differences	
	$f(Hz)$	$\bar{\gamma}$	$f(Hz)$	$\bar{\gamma}$	$\delta f(\%)$	$\delta\bar{\gamma}$
1	669.01	0.00	679.16	-1.487	1.52	-1.487
2	694.06	0.00	756.90	1.485	9.05	1.485
3	2172.6	0.00	2204.2	-1.521	1.45	-1.521
4	2249.2	0.00	2454.5	1.521	9.13	1.521
5	4541.7	0.00	4605.4	-1.546	1.40	-1.546

lution points are shifted by the length disturbance and the shifting is more steep in the higher modes. That is, the mode localization occurs in all modes simultaneously, and the degree of mode localization increases with the mode number. As shown in Fig. 3(b), the characteristic curves of the weakly coupled systems are close to the localization conditions. This means that if the coupling is very weak, many eigen solution points of the disordered system satisfy the localization condition, Eqs. (25) or (26), and the localization occurs to a large extent in those modes as shown in Fig. 4 and table 1 and 2. The degrees of mode localization are more severe in the weakly coupled systems than in strongly coupled ones. The effect of the coupling strength is more serious in lowest modes. The extents of the differences of each mode increase with the strength of the torsional spring, and the rate of the increase in the lowest mode is much higher than that in the higher modes as shown in table 1 and 2. Considering characteristic equation, Eq. (15), one can conclude that the first term of that equation can be neglected since β_1 and β_2 are very large in higher modes, and for this reason the mode localization of the higher modes are rarely affected by the coupling strength.

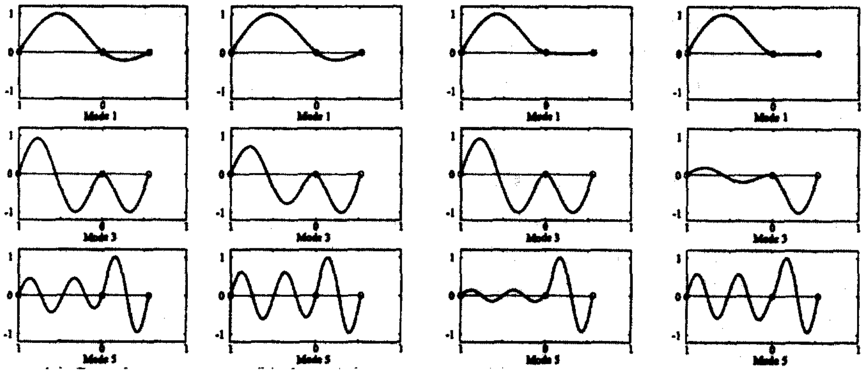
4.3. The Non-Periodic Structures; Cases 3 and 4.

The normal modes of the non-periodic two-span beams are examined for two coupling conditions. In the cases of the non-periodic systems, the words of ordered and disordered do not have meanings any longer because the undisturbed initial structures have no regularity now. So in this study the structures of case 3-a and case 4-a are called initial system, and the others disturbed system. The material properties, mass per unit length and flexural rigidity, of these cases are equal to those of the cases 1 and 2. The span lengths, however, are determined by considering the characteristic graph so that the mode localization may not occurs in the third mode satisfying $|\beta_1-\beta_2|=n\pi$ where $n=1$. Fig.



(a) case 3−a: α=0.5556; case 3−b: α=0.5278. (b) case 4−a: α=0.5556; case 4−b: α=0.5278.

Fig. 5. Characteristic Graph of the Case 3 and 4



(a) Case 3−a (b) Case 3−b (c) Case 4−a (d) Case 4−b

Fig. 6. Mode Shapes of the Case 3 and 4.

Table 3. Natural frequencies and degrees of mode localization: case 3.

Mode	Case 3-a		Case 3-b		Differences	
	$f(Hz)$	$\bar{\gamma}$	$f(Hz)$	$\bar{\gamma}$	δf(%)	$\delta\bar{\gamma}$
1	560.52	−0.7785	564.94	−0.7778	0.78	0.0007
2	1615.1	0.3878	1703.1	0.2031	5.45	−0.1847
3	2249.0	0.0000	2346.4	0.2273	4.33	0.2273
4	4289.1	−0.7223	4328.1	−0.7637	9.09	−0.0414
5	6331.6	0.6684	6771.0	0.3960	6.94	−0.2724

Table 4. Natural frequencies and degrees of mode localization: case 4.

Mode	Case 4-a		Case 4-b		Differences	
	$f(Hz)$	$\bar{\gamma}$	$f(Hz)$	$\bar{\gamma}$	δf(%)	$\delta\bar{\gamma}$
1	681.42	−2.889	681.48	−2.873	0.009	0.016
2	2142.8	0.511	2197.5	−0.925	2.55	−1.437
3	2249.0	−0.0000	2426.1	1.428	7.87	1.428
4	4615.1	−2.136	4616.9	−2.153	0.039	−0.017
5	7026.7	1.617	7703.7	0.425	9.63	−1.192

5 shows the characteristic graph of the cases 3 and 4. Selected mode shapes of each case are shown in Fig. 6.

The lowest five natural frequencies and the degrees of mode localization of each case and its differences are given in table 3 and 4.

The characteristic graphs depicted in Fig. 5 says that all the modes except third mode of initial system are already localized in some degrees, and Fig. 6 and table 3 and 4 confirm that. The length disturbance introduced into second span yields the changes in the natural frequencies and degrees of mode localization as predicted by the characteristic graphs, and Table 3 and 4 certify that. Observe that in some modes the degrees of mode localization are increased and in the others decreased by the disturbance. That is not in accordance with the result of the periodic system in which the trends of variations are consistent with the mode number in all modes. It is obvious, however, that the weak coupling makes the system sensitive to the disturbance independently of the periodicity of the system.

5. Conclusions

The present study has been concerned with the occurrence and the variation of the mode localization in both the periodic beams and the non-periodic ones, and investigated the effects of the coupling strength on the mode localization in those systems. The main findings of the work are summarized as follows.

(1) A mode satisfying $(\beta_1-\beta_2)=n\pi$ has possibility of drastic occurrence of mode localization independently of the periodicity of the two-span beams when the coupling is weak.

(2) Mode localization of a two-span beam with equal span lengths occurs simultaneously in all modes.

(3) Mode localization of a two-span beam with arbitrary span lengths occurs in some (not all) modes.

(4) Mode localization in higher modes is more sensitive to the system parameters than in lowest ones.

(5) The weak coupling makes the modes sensitive to mode localization. Especially this effect is more pronounced in the lowest modes than in the higher modes.

The second result is well known at the present time. However the others of the results are new ones or the extension of the already established works. From the results of this study, one can say that mode localization may occur in non-periodic structures as well as periodic ones. Extended studies on mode localization phenomena of general non-periodic structures are required.

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