비비례 감쇠시스템의 해석을 위한 효율적인 모드 중첩법

Efficient Mode Superposition Method for Non-classically Damped Systems

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국문요약

본 연구에서는 비비례 감쇠시스템을 효율적으로 해석할 수 있도록 모드 가속도법(mode acceleration method)과 모드 절삭 보강법(modal truncation augmentation method)을 확장하고 그 사용성을 검증하였다. 비례 감쇠시스템의 동응답해석에 널리 사용되는 모드 가속도법과 모드 절삭 보강법은 누락된 고차모드의 영향을 보정하여 모드 중첩법의 결과를 개선하는 방법이다. 기존의 방법들로 비비례 감쇠시스템을 해석하는 경우, 비비례 감쇠를 비례 감쇠로 근사하는 것이 일반적인 방법이다. 이러한 근사화 과정에서 필연적인 오차가 발생한다. 이에 본 연구에서는 구조물의 비비례 감쇠특성을 무시하지 않으며 정확하고 효율적으로 해석할 수 있도록 모드 가속도법과 모드 절삭보강법을 확장하였다. 비례 감쇠시스템에서는 모드 가속도법보다 모드 절삭 보강법이 더 효율적인 반면에, 비비례 감쇠시스템에서는 대부분의 경우에 있어서 확장된 두 방법의 효율성이 동일하다. 그러나 수치적 안정성은 확장된 모드 가속도법이 모드 절삭 보강법보다 우수하다. 이와 같은 확장된 모드 가속도법과 모드 절삭 보강법의 사용성 검증을 위해서 이론적 방법과 수치예제를 수행하였다.

주요어 : 모드 중첩법, 비비례 감쇠, 모드 가속도법, 모드 절삭보강법

ABSTRACT

The mode acceleration and the modal truncation augmentation methods are expanded to non-classically damped systems. The mode acceleration and the modal truncation augmentation methods improve the standard mode superposition method by complementing the portion of the truncated high modes. These methods have been used only in classically damped system. To use these methods, non-classically damped system has been approximated to the classically damped system. However, in this paper, the mode acceleration and the modal truncation augmentation methods are expanded to non-classically damped system for the efficient and accurate analysis without the approxi- mations. The state space approach is used for the expansion of each method. The applicability of expansion is verified analytically and numerically. The expanded modal truncation augmentation method is conditionally stable depending on the external loading whereas the expanded mode acceleration method is unconditionally stable. The stability condition for the expanded modal truncation augmentation method is suggested. In the stable case, the results of the expanded modal truncation augmentation method are the same with those of the expanded mode acceleration method. Two numerical results are presented to validate the analytical conclusions about the characteristics of the expanded mode acceleration and the modal truncated augmentation methods.

Key words : *mode superposition method, non-classically damping, mode acceleration method, modal truncation augmentation method*

1. Introduction

The methods of dynamic analysis are divided into the direct integration method and the mode superposition method. The direct integration method is used for the short duration loading as an impulse and the mode superposition method is used for the long duration loading as an earthquake. Unfavorably, the

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본 논문에 대한 토의를 2000년 6월 30일까지 학회로 보내 주시면 그 결과를 게재하겠습니다.

reduction process of the mode superposition method alters the modal representation of loading and can adversely affect the quality of the calculated responses. The mode acceleration method(MA method) and the modal truncation augmentation method(MT method) improve the standard mode superposition method by complementing the truncated high modes. The MA method was originally proposed by Williams[1] and the MT method was suggested by Dickens and Wilson.[2] Dickens et al[3] compared both methods and well documented it.

In dynamic analysis of linear systems, it is common to assume that the system is classically damped and the MA and MT methods have been used only in classically damped system. In practice, non-classically damped systems can be approximated by a classically damped systems by neglecting the coupled damping terms.[4] However, there are importance situations such as soil-structure interaction systems, composite structures or etc., where the effect of non-classically damping is essential and must be included in the analysis.

In this paper, the MA and MT methods are expanded to non-classically damped system for the efficient and accurate analysis without the approximations. The state space approach is used for the expansion of each method. The state space approach[5],[6] is a general method of dynamic analysis of multi-degree-of-freedom dynamic systems, which is applicable to the systems with classically and non-classically damping not neglecting the coupled damping. The applicability of expansion is verified by the closed form solutions and the numerical examples.

The first section of this paper gives a simple overview of the standard mode superposition method for classically damped systems and non-classically damped systems. The second and third sections describe the expanded MA and MT methods for the non-classically damped systems, respectively. In the fourth section, the characteristic of the expanded MT method is described and the stability condition for the expanded MT method is suggested. In the last section, the analytical conclusions are validated by the two numerical examples.

2. Mode Superposition Method for Non-classically Damped Systems

Let the discrete equations of motion of a linear structural system be given by

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{C}\dot{\mathbf{u}}(t)+\mathbf{K}\mathbf{u}(t)=\mathbf{R}_0 r(t) \quad (1)$$

where **M**, **C** and **K** are the n by n mass, damping and stiffness matrices, respectively; and $\ddot{\mathbf{u}}(t)$, $\dot{\mathbf{u}}(t)$ and $\mathbf{u}(t)$ are the, n by 1 acceleration, velocity and displacement vectors, respectively; the applied loading is composed of two parts: $\mathbf{R}_0$ is the invariant spatial portion and $r(t)$ is the time varying portion.

For a modal response analysis, the physical coordinates of equation (1) are transformed to modal coordinates, $q(t)$, by a retained set of eigenvectors of the system, $\mathbf{\Phi}$

$$\mathbf{u}(t)=\mathbf{\Phi}q(t) \quad (2)$$

where $\mathbf{\Phi}$ are determined from the general eigenvalue problem

$$\mathbf{K}\mathbf{\Phi}=\mathbf{M}\mathbf{\Phi}\Omega^2 \quad (3)$$

where Ω^2 are eigenvalues corresponding to the eigenvectors: $\Phi=[\phi_1\ \phi_2 \cdots \phi_n]$.
If the system is classically damped, equation (2) is used to transform equation (1) to n

decoupled equations of modal coordinates

$$\ddot{q}_i + 2\beta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \phi_i^T \mathbf{R}_0 r(t) \quad i = 1, 2, \cdots, n \tag{4}$$

where ϕ_i, ω_i and β_i are the i th eigenvector, natural frequency and damping ratio of the system.

The mode displacement (MD) method, the standard procedure for determining responses, consists of expanding the modal responses, solved from equation (4), into approximate physical responses using the retained m modes[7],[8]

$$\mathbf{u}(t) = \sum_{i=1}^{m} \phi_i q_i(t) \qquad (m << n) \tag{5}$$

where n is the total degree of the freedom of system and m is number of retained modes. The MD method gives approximated responses because m, the number of retained modes, is much smaller than n, the total degree of the freedom of the system.

If the system is non-classically damped, equation (1) can not be transformed to decoupled equations because the damping matrix, **C** is not diagonalized. To apply mode superposition method to non-classically damped system, the second-order differential equation. (1) can be transformed into a first-order differential equation by doubling the size of the system, which is the state space approach

$$\mathbf{B}\dot{\mathbf{y}}(t) - \mathbf{A}\mathbf{y}(t) = \mathbf{F}_0 r(t) \tag{6}$$

where **A** and **B** is the $2n$ by $2n$ matrix composed of **M**, **C** and **K** and y(t) is the $2n$ by 1 vector

$$\mathbf{B} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \quad \mathbf{y}(t) = \begin{Bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{Bmatrix}$$

$$\mathbf{F}_0 = \begin{Bmatrix} \mathbf{R}_0 \\ 0 \end{Bmatrix} \tag{7}$$

where **0** is the n by 1 zero vector.

For a modal response analysis, the physical coordinates of equation (6) are transformed to modal coordinates, z(t), by a retained set of eigenvectors of the system, $\mathbf{\Psi}$

$$\mathbf{y}(t) = \mathbf{\Psi} z(t) \tag{8}$$

where $\mathbf{\Psi}$ are determined from the eigenvalue problem

$$\mathbf{A}\psi_i = s_i \mathbf{B}\psi_i \tag{9}$$

where ψ_i, i th eigenvector, and s_i, corresponding eigenvalue, may occur in complex-conjugate pairs, respectively.

For non-classically damped systems, equation (6) can be transformed to $2n$ decoupled modal equations as in equation (10) by using the modal transformation of equation (8).

$$\dot{z}_i - s_i z_i = \psi_i^T \mathbf{F}_0 r(t) \qquad i = 1, 2, \cdots, 2n \tag{10}$$

The mode displacement (MD) method for non-classically damped systems can give approximate physical responses using retained $2q$ modes

$$\mathbf{y}(t) = \sum_{i=1}^{2q} \psi_i z_i(t) \qquad (q << n) \tag{11}$$

where n is the degree of the freedom of original system and $2q$ is number of retained modes.

3. Mode Accleration Method for Non-classically Damped Systems

For the modal response analysis, the mode superposition method generally uses the lower $2q$ modes among the total $2n$ modes as in equation (11). The truncated high modes which are not retained in the modal response analysis, induce the errors of approximation.[4] The MA and MT methods reduce the error efficiently by complementing the effect of truncated high modes without generating the additional modes except $2q$ modes.

But the MA and MT methods were used only in classically damped systems because of the limitations of non-classically damped systems such as difficulty of diagonalization and the efficiency of generating the eigenvectors. The MA and MT methods are newly expanded to the non-classically damped systems by using the state space approach.

To expand the MA algorithm, equation (4) is rewritten as in equation (12).

$$\mathbf{y}(t) = \mathbf{A}^{-1}\mathbf{B}\dot{\mathbf{y}}(t) - \mathbf{A}^{-1}\mathbf{F}(t) \tag{12}$$

Using equation (11), the displacements by the expanded MA algorithm, $\mathbf{y}_{ma}(t)$, are given by.

$$\mathbf{y}_{ma}(t) = \mathbf{A}^{-1}\mathbf{B}\sum_{i=1}^{2q}\psi_i\dot{\mathbf{z}}_i - \mathbf{A}^{-1}\mathbf{F}(t) \quad (q << n) \tag{13}$$

The responses by the expanded MA method consists of two parts as mentioned by Dickens and et al[3] in classically damped systems.

$$\mathbf{y}_{ma}(t) = \mathbf{y}_s(t) + \mathbf{y}_{t_{ma}}(t) \tag{14}$$

The first portion, $\mathbf{y}_s(t)$, is the displacements obtained from the retained modes and so same with the displacements of the MD method given in equation (11). The second part, $\mathbf{y}_{t_{ma}}(t)$, is the displacements lost due to the modal truncation, which is given by subtracting equation (11) from equation (13).

$$\mathbf{y}_{t_{ma}} = -\mathbf{A}^{-1}\mathbf{R}_t r(t) \tag{15}$$

Equation (15) is the portion of the solution not represented by the modes retained in the analysis, which complements the effect of truncated high modes. In the equation (15), $\mathbf{R}_t$ is force truncation vector defined as

$$\mathbf{R}_t = \mathbf{F}_0 - \mathbf{R}_s \tag{16}$$

where $\mathbf{R}_s$ is the modally represented spatial load vector

$$\mathbf{R}_s = \mathbf{B}[\,\psi_i\ \overline{\psi}_i\,][\psi_i\ \overline{\psi}_i\,]^T\mathbf{F}_0 \tag{17}$$

where ψ_i and $\overline{\psi}_i$ are the eigenvectors in complex-conjugate pairs. Modal-physical representation of displacements and loads like $\mathbf{R}_s$ and $\mathbf{R}_t$, are well documented by J. M. Dickens, J. M. Nakagawa and M. J. Wittbrodt.[3] But in this paper, the row size of each vector, $\mathbf{R}_s$ and $\mathbf{R}_t$, is doubled not same with the system matrix, **M**, **C** and **K** because the state space approach is used. $\mathbf{R}_s$ for the non-classically damped system is calculated from complex-conjugate pairs of eigenvector because $\mathbf{R}_s$ has physical meaning of modally represented spatial load vector which can not be in the complex domain.

4. Modal Truncation Augmentaion Method for Non-classically Damped Systems

The responses by the MT method consist of two parts as the MA method.

$$\mathbf{y}_{mt}(t) = \mathbf{y}_s(t) + \mathbf{y}_{t_{mt}}(t) \tag{18}$$

The first portion, $\mathbf{y}_s(t)$, is the displacements obtained from the retained modes and so same with the displacements of the MD method given by equation (11). The second part, $\mathbf{y}_{t_{mt}}(t)$, is the displacements lost due to the modal truncation. The MT method approximates the non-modally represented solution by the MT vector, $\mathbf{P}$ which complements the effect of truncated high modes. A MT vector is determined by solving for the displacement vector, $\overline{\mathbf{P}}$, due to the force truncation vector of equation (16).

$$\mathbf{A}\overline{\mathbf{P}} = \mathbf{R}_t \tag{19}$$

The MT vector, P, can be formed

$$\mathbf{P} = \frac{1}{\alpha}\overline{\mathbf{P}} \tag{20}$$

where $\alpha = (\overline{\mathbf{P}}^T\mathbf{B}\overline{\mathbf{P}})^{1/2}$. The MT vector has a mathematical consistence with Rayleigh-Ritz approximation where the assumed Ritz basis vectors are derived using the spatial force truncation vector. The MT vector is orthogonal on the matrix $\mathbf{A}$ and $\mathbf{B}$ but do not satisfy the eigenvalue problem at each equation.

For a modal response analysis by the MT method, the transformation is used.

$$\mathbf{y}_{t_{mt}} = \mathbf{P}z_P(t) \tag{21}$$

Using this transformation to reduce equation (6).

$$\mathbf{P}^T\mathbf{B}\,\mathbf{P}\,\dot{z}_P(t) - \mathbf{P}^T\mathbf{A}\,\mathbf{P}\,z_P(t) = \mathbf{P}^T\mathbf{R}_t r(t) \tag{22}$$

Equation (22) is rewritten to be solved for $z_P(t)$

$$\dot{z}_P(t) - s_P z_P(t) = \mathbf{P}^T\mathbf{R}_t r(t) \tag{23}$$

where $s_p = \mathbf{P}^T\mathbf{A}\mathbf{P}$. After $z_P(t)$ is calculated from equation (23), $z_P(t)$ can then be back transformed using equation (21) to yield the MT solution which complements the non-retained modes.

The MA and MT methods both approximate the non-modally represented solution to complement truncated high modes. In classically damped systems, it would be expected that the MT method would give better results overall than the MA method because of added dynamics.[8] In non-classically damped systems, however, the MT method has different characteristics with in classically damped systems. The characteristics of the MT method for non-classically damped systems are discussed in two points, the stability of the solution and the characteristics of the solution when the solution is stable.

5. The Charateristics of the Modal Truncation Augmentaion Method For Non-classically Damped Systems

5.1 The Stability

To check the stability of the MT method, the general solution of equation (24) is solved for the initial condition, $z_P(0) = 0$, when the harmonic force, $r(t) = sin(\varpi t)$, is applied.

$$z_P(t) = \frac{\mathbf{P}^T\mathbf{R}_t}{s_P^2 + \varpi^2}\left(\varpi e^{s_P t} - s_P\, sin(\varpi t) - \omega\, cos(\varpi t)\right) \tag{24}$$

It is appropriate to assume the applied force to be harmonic function because the applied force like an earthquake can be expressed by combining the harmonic functions.

Equation (24) is stable when the $s_p = \mathbf{P}^T\mathbf{A}\mathbf{P}$ satisfies the stability condition:

$$s_p = \mathbf{P}^T\mathbf{A}\mathbf{P} < 0 \tag{25}$$

If the condition of equation (25) is not satisfied, the MT solution would be divergent as time goes on. Therefore, the MT method is stable and applicable to modal response analysis for non-classically damped systems when the stability condition of equation (25) is satisfied. Because the matrix **A** is generally indefinite[9], the stability of the solution by the MT method depends on the MT vector, **P** formed from the equation (19) and (20), which means that stability of MT method is affected by input load.

5.2 The Characteristics of the Solution by MT Method

The matrix **A** consists of mass matrix, **M**, and stiffness matrix, **K** and so the elements of the matrix **A** have the value of the material properties of the structure. On the other hand, the frequency of input loading, $\overline{\omega}$, has the relatively small value of range from several tens to several hundreds in rad/sec. Therefore, generally, the absolute value of $s_p = \mathbf{P}^T\mathbf{A}\mathbf{P}$ is much bigger than the $\overline{\omega}$.

$$|s_p| >> \overline{\omega} \tag{26}$$

Using the condition of equation (26), Equation (24) can be simplified.

$$z_P(t) \approx \frac{-\mathbf{P}^T\mathbf{R}_t}{s_p} sin(\overline{\omega}\, t) \tag{27}$$

Then $z_P(t)$ can be back transformed by using equation (21) to yield the MT solution for the displacements which present the non-retained modes.

$$\mathbf{y}_{t_{mt}} = \mathbf{P}z_P(t) = -\frac{\mathbf{P}\mathbf{P}^T\mathbf{R}_t}{s_P} sin(\overline{\omega}\, t) \tag{28}$$

Using the relation of equation (19) and (20), the MT vector, **P**, can be represented as follows.

$$\mathbf{P} = \frac{1}{\alpha}\mathbf{A}^{-1}\mathbf{R}_t \tag{29}$$

Substituting equation (29) into equation (28), the displacements by MT solution is as follows.

$$\mathbf{y}_{t_{mt}} = -\mathbf{A}^{-1}\mathbf{R}_t \frac{\mathbf{P}^T\mathbf{R}_t}{\alpha\, s_p} sin(\overline{\omega}\, t) \tag{30}$$

To be compared with MA solution in equation (15), the MT solution is similar except the coefficient. If the previous relations such as $\hat{\mathbf{R}}_t = \alpha\mathbf{A}\mathbf{P}$ and $s_p = \mathbf{P}^T\mathbf{A}\mathbf{P}$ are used, the MT solution is simplified and identical with the MA solution.

$$\mathbf{y}_{t_{mt}} = -\frac{\mathbf{P}^T\hat{\mathbf{R}}_t}{\alpha\, s_p}\mathbf{A}^{-1}\hat{\mathbf{R}}_t\, sin(\overline{\omega}\, t) = -\mathbf{A}^{-1}\hat{\mathbf{R}}_t\, sin(\overline{\omega}\, t) = \mathbf{y}_{t_{ma}} \tag{31}$$

In summary, the applicability of the expanded MT method is limited by the stability condition in equation (25). When the stability condition is satisfied, the MT method is stable and applicable to modal response analysis for non-classically damped systems but the result of the MT solution is identical with the MA solution as proved in equation (26)-(31).

6. Numerical Examples

To verify the applicability of the expanded MA and MT methods, the numerical examples are carried out. In the first numerical example, the efficiency of the MA and MT methods is compared with the MD method. In the second numerical example, both convergent case and divergent case of the MT method are presented. In the stable case of the MT method, it is shown that the result is identical with the MA method.

For the comparison with accurate solution, each result is normalized by the result of the direct integration method which is exact solution of the system.

6.1 Cantilever Beam Subjected to Earthquake

This numerical example shows that the expanded MA and MT methods are more efficient than the MD method as in classically damped systems. The system shown in Fig. 1 is a cantilever beam with a lumped translational viscous-damper attached at each node. The cantilever beam is modeled by 10 equal finite elements deduced from elementary beam theory. The Youngs modulus, area, inertia and density of beam are respectively 3.0×10^7psi, $4in^2$, $1.25in^4$ and $7.41\times10^{-4}lb/in^3$. The damping coefficient of the tangential damper is 0.1lb·sec/in. The load applied to system is El-Centro earthquake. Table 1 shows the complex-conjugate pairs of eigenvalue.

The results of the response analysis are shown from Fig. 2 to Fig. 5. In the legend of the figure, md1, ma1 and mt1 means that the lowest one eigenvector is used for the modal response analysis by the MD, MA and MT method, respectively. md2, ma2 and mt2 means that the lowest two eigenvectors are used in the modal response analysis by each method.

Fig. 2 and Fig. 3 show that the MA and MT solutions for the moment at each node. In these figures, the MA and MT methods are more efficient than the MD method because the MA and MT methods contained the non- modally represented portion besides the modally represented portion of the MD method. Fig. 4 and Fig. 5 show that the MA and MT solutions for the shear force at each node. In these figures, the MA and MT methods are more efficient than the MD method.

Through the above numerical example, it is shown that the expanded MA and MT methods are well applied to a non-classically damped systems and the results has same trend with classically damped system in which the MA and MT methods are more efficient. In classically damped systems, it would be expected that the MT method would give better results overall than the MA method because of added dynamics.[8] In non-classically damped systems, however, the MT method gives the same results with the MA method when the MT method is stable.

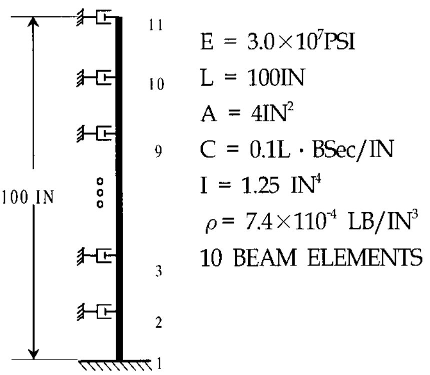

Fig. 1 Beam configuration

Table 1 Eigenvalue set

Mode number	Eigenvalues
1	−4.43482 − 39.29620i
2	−4.43482 + 39.29620i
3	−88.4454 − 231.3995i
4	−88.4454 + 231.3995i
5	−677.3535 − 147.892i
6	−677.3535 + 147.892i

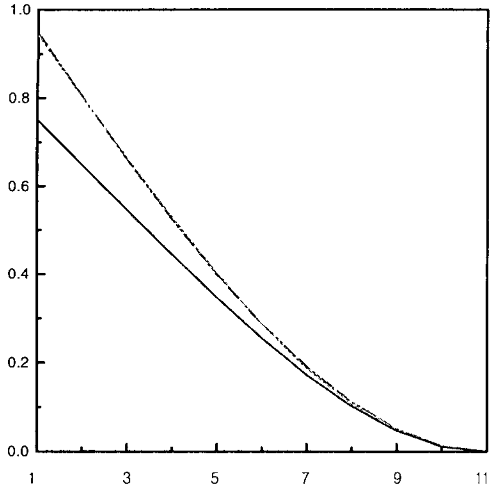

Fig. 2 Moment by the MD method

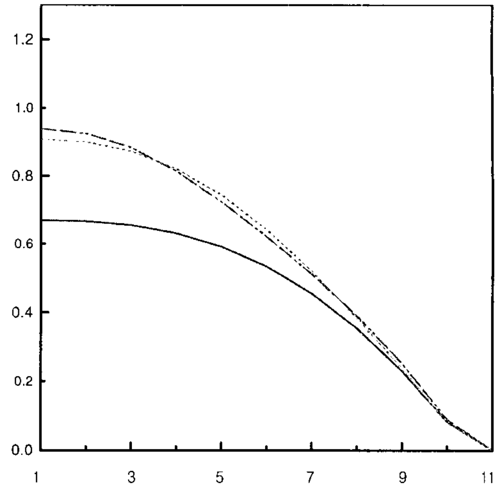

Fig. 4 Shear by the MD method

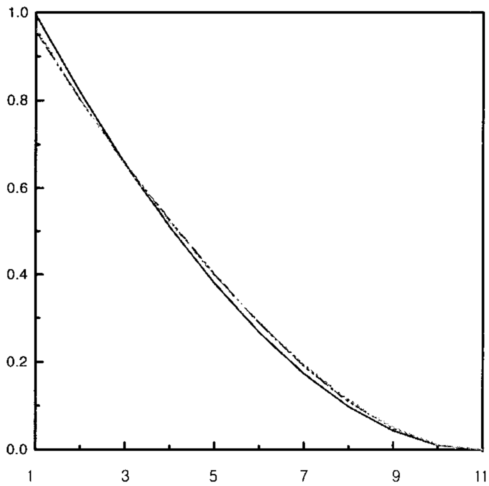

Fig. 3 Moment by the MA and MT methods

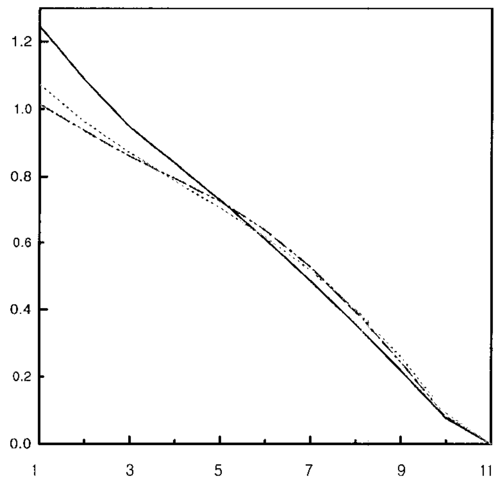

Fig. 5 Shear by the MA and MT methods

6.2 10-Story Shear Building

This numerical example shows both convertgent and divergent cases of the MT method. The structure shown in Fig. 6 is non-classically damped system, which has 10 degrees of freedom with a lumped translational viscous-damper attached at seventh floor. The complex-conjugate pairs of the eigenvalue are presented in Table 2. The responses by the MA and MT methods are calculated when load case 1 and load case 2 are applied respectively and the results of two methods are compared. The time varying portion of input load, $sin(\overline{\omega} t)$, is identical for both cases. The frequency of applied load is 32.0rad/sec.

The displacements for the load case 1 are presented in Fig. 7 and Fig. 8. For the load case 1, the value of $s_p = 9.4710 \times 10^3$ satisfies the stability condition, $s_p = \mathbf{P}^T \mathbf{A} \mathbf{P} < 0$. Because the stability condition is satisfied, the expanded MT method is stable as in Fig. 7 and gives same result with the expanded MA method in Fig. 8.

The displacements for the load case 2 are presented in Fig. 9 and Fig. 10. For the load case 2, the value of $s_p = 6.2443 \times 10^3$ do not satisfied the stability condition, $s_p = \mathbf{P}^T \mathbf{A} \mathbf{P} < 0$. Because the stability condition is not satisfied, the MT method is divergent and gives no solution as in Fig. 9. The MA solution, whereas, is stable and gives the result as in Fig. 10.

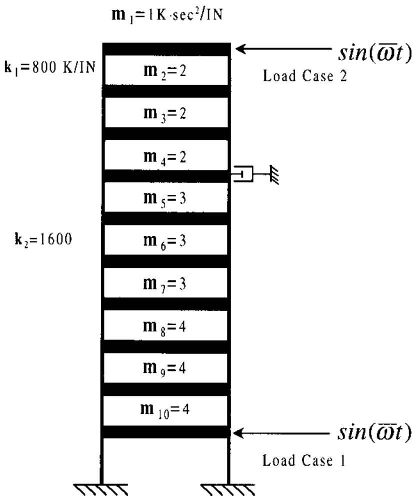

Fig. 6 Shear building configuration

Table 2 Eigenvalue set

Mode Number	Eigenvalues
1	-0.0316 - 4.0100i
2	-0.0316 + 4.0100i
3	-0.0066 - 10.8381i
4	-0.0066 + 10.8381i
5	-0.0058 - 17.421i
6	-0.0058 + 17.421i

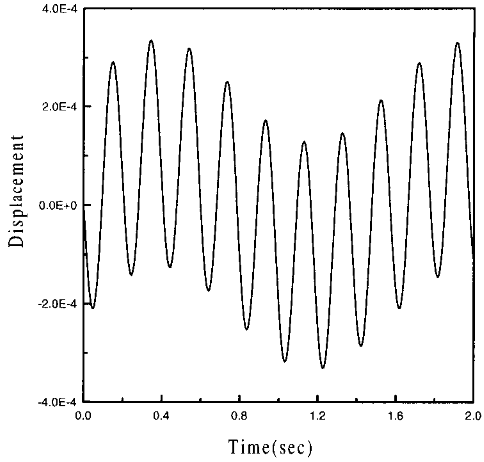

Fig. 7 Response by the MT method

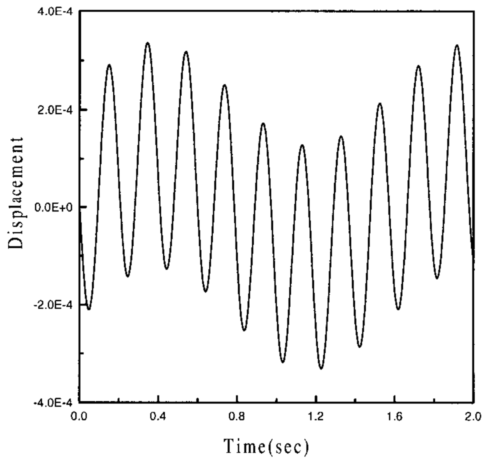

Fig. 8 Response by the MA method

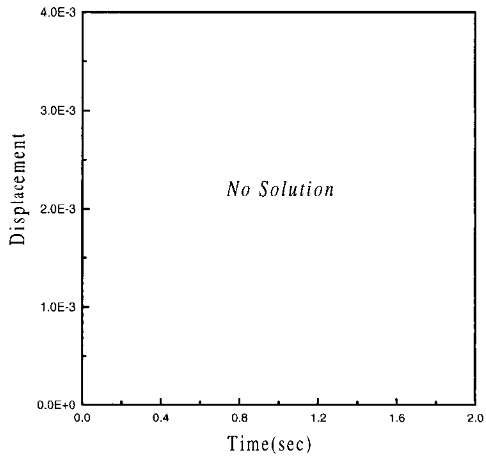

Fig. 9 Response by the MT method

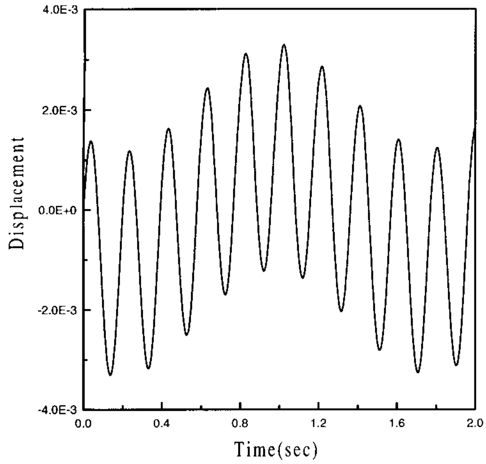

Fig. 10 Response by the MA method

From the this numerical example, it can be said that the applicability of the expanded MT method is limited by the stability condition and when the stability condition is satisfied, the MT method is stable and gives same result with the expanded MA solution as shown in Fig. (7)~(10).

7. Conclusions

In this paper, the MA and MT methods were expanded to the non-classically damped systems for an efficient modal response analysis. The applicability of expansion was verified analytically and numerically.

The expanded MA and MT methods are more efficient than the standard MD method, which is the same tendency with classically damped systems. The expanded MT method is conditionally stable depending on the external loading whereas the expanded MA method is unconditionally stable. The stability condition, $s_p = \mathbf{P}^T\mathbf{A}\mathbf{P} < 0$ for the expanded MT method was suggested. In the stable case, the results of the MT method are the same with those of the MA method. Therefore the expanded MA method is better than the expanded MT method for an analysis of the non-classically damped systems.

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