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Application of Matrix-Powered Lanczos Algorithm to Eigensolution Method for Structures

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Abstract

This paper comprehensively investigates the applicability of the modified Lanczos method using the power technique, which was developed in the field of quantum physics, to the eigenproblem in the field of engineering mechanics by rigorously deriving the modified Lanczos recursion procedure considering power of matrix and numerically evaluating the suitable power value. The matrix-powered Lanczos method has not only the better convergence but also the less operation count than the conventional Lanczos method because the matrix-powered Lanczos method can reduce the required number of Lanczos vectors. However, since increasing power value in the method may cause numerical instability, resulting in failure in convergence, special care must be taken in the selection of appropriate power. By analyzing four numerical examples with different sizes of system, the effectiveness of the matrix-powered Lanczos method is verified and the appropriate power is also recommended.

Keywords: *eigenproblem of structures, matrix-powered Lanczos method, suitable power of the dynamic matrix*

1. Introduction

Eigenvalue analysis is an important step in structural dynamic analysis when the mode superposition method is used. Many solution methods have been developed for eigenvalue analysis, and among these methods the Lanczos method has been known to be very efficient for solving large eigenvalue problems (Hughes, 1987). The Lanczos method was originally developed for evaluating the eigensolution of matrices through the Rayleigh-Ritz reduction of the eigensystem to tridiagonal form (Lanczos, 1950). In his original paper, Lanczos described the algorithm as a method for evaluating eigenvalues and the corresponding eigenvectors of a general matrix. The eigenvectors are constructed by forming a linear combination of a set of vectors, known as Lanczos vectors, computed in the course of the algorithm. Many researches in the past years have studied the Lanczos process, and the algorithm has been extended to symmetric generalized eigenproblems. It is now widely accepted as the method of solution of eigenproblems.

To improve the Lanczos method many researchers have studied a variety of procedures such as shifting technique, restarted algorithm and conjugate gradient scheme. The shifting technique was proposed by Erricson and Ruhe (1980) to accelerate the Lanczos algorithm and they showed that the accelerated method can be applied effectively to the solution of generalized symmetric eigenproblems in which the matrices are large and sparse. Grimes *et al.* (1994) applied the shifting technique to the block Lanczos algorithm. Ruhe (1998) extended the shifted Lanczos method to large nonsymmetric eigenproblems. Recently, Komzsik (2001) modifed NASTRAN algorithm by using the technique of shift to solve quadratic eigenvalue problems. Smith *et al.* (1993) have accelerated the Lanczos method through an implicitly restarted technique and applied to frequency dependent and nonlinear eigenproblems. Benner and Faßbender (1997) presented an implicitly restarted symplectic Lanczos method for the Hamiltonian eigenvalue problem. Zhang (1998) improved the convergence of the Lanczos method by combining explicitly restarted technique and Davidson algorithm. Gambolati and Putti (1994) employed the preconditioned conjugate gradi-

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ent scheme in the Lanczos method and compared the modified Lanczos method with optimization method. They concluded that the modified Lanczos method is faster for small systems, while optimization method is better for large systems.

In the field of quantum physics, Grosso *et al.* (1993) modified the Lanczos algorithm with the power of operator to obtain the eigenstate of quantum systems. The technique of power of operator has been adopted by Cordelli (1994), Grosso *et al.* (1995), Bevilacqua *et al.* (1996) and Fornari *et al.* (1997). In the field of engineering mechanics, the accelerated subspace iteration method used similar power technique to more efficiently calculate eigenvalues and corresponding eigenvectors (Lam and Bertolini, 1994; Qian and Dhatt, 1995; Bertolini and Lam, 1998; Wang and Zhou, 1999). The power technique is applied to the simultaneous inverse iteration process. On the other hand, the modified Lanczos method using the power technique is not applied yet.

This paper rigorously derives the modified Lanczos recursion using the power technique, which was not presented in the previous studies in quantum physics, and first applies it to the eigenproblem in engineering mechanics. The power technique can be applied to the matrix $K^{-1}M$. The matrix $K^{-1}M$ is called the dynamic matrix (Clough and Penzien, 1993). Increasing power value may cause numerical instability, resulting in failure in convergence. Therefore, special care must be taken in the selection of power value. This paper numerically evaluates the appropriate power of the dynamic matrix, which was not discussed in previous studies. Four numerical examples are analyzed to verify the effectiveness of the matrix-powered Lanczos method. The suitable power of the dynamic matrix in the method is also recommended through numerical examples.

2. Matrix-Powered Lanczos Method

In the field of quantum physics, the following Lanczos recursion is used to obtain the eigenstate of quantum systems.

$$b_{n+1}f_{n+1} = Hf_n - a_nf_n - b_nf_{n-1} \tag{1}$$

where H is a given operator, f is basis functions, a and b are coefficients and n is Lanczos step number. In the eigenproblem of engineering mechanics, H and f correspond to the dynamic matrix and Lanczos vectors, respectively. Grosso *et al.* (1993) modified equation (1) by introducing the second power of operator to accelerate the convergence. The Lanczos recursion modified by them are as follows:

$$b_{n+1}f_{n+1} = (H-E_t)^2 f_n - a_nf_n - b_nf_{n-1} \tag{2}$$

where E_t is trial energy which corresponds to shift.

The concept of power technique in equation (2) can be applied to the eigenproblem in engineering mechanics, and then the modified Lanczos method can be derived as follows.

2.1 Modified Lanczos Recursion

The eigenproblem of structures frequently encountered in engineering mechanics can be expressed as

$$K\phi_i = \lambda_i M\phi_i (i = 1, 2, \ldots, n) \tag{3}$$

where K and M are symmetric stiffness and mass matrices of order n, respectively. λ_i and ϕ_i are the ith eigenvalue and associated eigenvector of the system.

The Lanczos algorithm is equivalent to obtaining Ritz bases vectors through Gram-Schmidt orthogonalization of the Krylov sequence as follows (Hughes, 1987):

$$x_{i+1} = v_i - \sum_{j=1}^{i} \upsilon_j x_j \tag{4}$$

with

$$v_i = (K_\mu^{-1}M)^i x_0 \tag{5}$$

where v_i is Krylov sequence, x_0 is a starting vector, x_j is jth Lanczos vector, υ_j is the component of v_i along x_j, $K_\mu = K - \mu M$ and μ is shift. The concept of power technique can be applied to the dynamic matrix in the Krylov sequence, then the following modified Gram-Schmidt orthogonalization of the Krylov sequence can be introduced.

$$x_{i+1} = v_i - \sum_{j=1}^{i} \upsilon_j x_j \tag{6}$$

with

$$v_i = ((K_\mu^{-1}M)^\delta)^i x_0 \tag{7}$$

where v_i is modified Krylov sequence, δ is positive integer. The term, $(K_\mu^{-1}M)^\delta$, means the power of the dynamic matrix. Eqs. (6) and (7) mean that an approximated eigenvector, whose number of iteration is δi, is contained in $(i+1)$ Lanczos vectors. On the other hand, in Eqs. (4) and (5), $(i+1)$ Lanczos vectors contain an approximated eigenvector whose number of iterations is i. Therefore, Lanczos vectors in equation (6) give a better solution than those in equation (4).

To derive the matrix-powered Lanczos recursion, assume that the first i Lanczos vectors are founded, and the next Lanczos vector, x_{i+1}, will be constructed. Because Lanczos vectors are resultant vectors from Gram-Schmidt orthogonalization process, all the Lanczos vectors satisfy following orthonormal condition

$$x_i^T M x_j = \delta_{ij} \tag{8}$$

where δ_{ij} is the Kronecker delta. From the definition of modified Kryolv sequence, equation (7) can be rearranged by

$$v_i = (K_\mu^{-1} M)^\delta v_{i-1} \tag{9}$$

and, $(i-1)$th Kryolv sequence has the following form according to equation (6)

$$v_{i+1} = \sum_{j=1}^{i} \hat{\upsilon}_j x_j \tag{10}$$

Substituting equation (10) into equation (9), we obtain

$$v_i = \sum_{j=1}^{i} \hat{\upsilon}_j (K_\mu^{-1} M)^\delta x_j \tag{11}$$

By equation (11), $(i-1)$th Kryolv sequence has another form as

$$v_{i-1} = \sum_{j=1}^{i-1} \hat{\upsilon}_j (K_\mu^{-1} M)^\delta x_j \tag{12}$$

From equations (10) and (12), we can obtain following relation

$$\sum_{j=1}^{i} \hat{\upsilon}_j x_j = \sum_{j=1}^{i-1} \hat{\upsilon}_j (K_\mu^{-1} M)^\delta x_j \tag{13}$$

Equation (13) means that each $(K_\mu^{-1} M)^\delta x_j$ can be written as a linear combination of the first $(j+1)$ Lanczos vectors. Therefore, equation (11) can be rewritten by

$$v_i = \hat{\upsilon}_i (K_\mu^{-1} M)^\delta x_i + \sum_{j=1}^{i-1} \hat{\upsilon}_j (K_\mu^{-1} M)^\delta x_j$$

$$= \hat{\upsilon}_i (K_\mu^{-1} M)^\delta x_i + \sum_{j=1}^{i} \bar{\upsilon}_j x_j \tag{14}$$

Substituting equation (14) into equation (6), then equation (6) becomes

$$x_{i+1} = \hat{\upsilon}_i (K_\mu^{-1} M)^\delta x_i - \sum_{j=1}^{i} (\upsilon_j - \bar{\upsilon}_j) x_j \tag{15}$$

Equation (15) means that the orthogonalization of Krylov sequence is equivalent to that of $(K_\mu^{-1} M)^\delta x_i$. Therefore, next Lanczos vector x_{i+1} is obtained by first computing preliminary vectors $\bar{x}_i$ as

$$\bar{x}_i = (K_\mu^{-1} M)^\delta x_i \tag{16}$$

From equation (15), these preliminary vectors can be expressed as

$$\bar{x}_i = \tilde{x}_i + \alpha_i x_i + \beta_{i-1} x_{i-1} + \gamma_i x_{i-2} + \dots \tag{17}$$

with

$$\tilde{x}_i = \tilde{\beta}_i x_{i+1} \tag{18}$$

where $\alpha_i, \beta_{i-1}, \gamma_i$ and $\tilde{\beta}_i$ are scalar coefficients. The coefficient α_i can be obtained by premultiplying both sides of equation (17) by $x_i^T M$. Then using the orthogonal relationships, equation (8), following result is derived.

$$\alpha_i = x_i^T M \bar{x}_i \tag{19}$$

The component β_{i-1} may be obtained similarly by premultiplying $x_{i-1}^T M$. Then using equation (8), β_{i-1} is calculated by

$$\beta_{i-1} = x_{i-1}^T M \bar{x}_i \tag{20}$$

Introducing equation (16) into equation (20), we can get

$$\beta_{i-1} = x_{i-1}^T M (K_\mu^{-1} M)^\delta x_i = x_{i-1}^T (M K_\mu^{-1})^\delta M x_i \tag{21}$$

and applying the transpose of equation (16) to x_{i-1}^T, we can get

$$\beta_{i-1} = \bar{x}_{i-1}^T M x_i \tag{22}$$

Finally, expanding $\bar{x}_{i-1}$ in terms of equation (17), and then using equation (8), we can get the following result

$$\beta_{i-1} = \tilde{x}_{i-1}^T M x_i \tag{23}$$

or rewriting β_i for the $(i+1)$st vector

$$\beta_i = \tilde{x}_i^T M x_{i+1} \tag{24}$$

Substituting equation (18) into equation (24), following relations are obtained.

$$\beta_i = \tilde{\beta}_i x_{i+1}^T M x_{i+1} = \tilde{\beta}_i \tag{25}$$

From equations (18) and (25)

$$x_{i+1} = \frac{\tilde{x}_i}{\beta_i} \tag{26}$$

Introducing equation (26) into equation (24), we can calculate β_i by

$$\beta_i = (\tilde{x}_i^T M \tilde{x}_i)^{1/2} \tag{27}$$

Continuing in the same ways as for finding the expression in equation (23), the coefficient γ_i is obtained to be

$$\gamma_i = \tilde{x}_{i-2}^T M x_i \tag{28}$$

Substituting equation (18) for $\tilde{x}_{i-2}$ in equation (28), we can obtain

$$\gamma_i = \beta_{i-2} x_{i-1}^T M x_i = 0 \tag{29}$$

A corresponding procedure could be used to demonstrate that all further terms in the expansions for x_i will be zero. Therefore, equation (17) can be rewritten as the three-term recursive formulas as

$$\tilde{x}_i = (K_\mu^{-1} M)^\delta x_i - \alpha_i x_i - \beta_{i-1} x_{i-1} \tag{30}$$

2.2 Reduction to Modified Tridiagonal System

After q steps, we have a set of Lanczos vectors, $X=[x_1\ x_2\ \cdots\ x_q]$. Here, we employ the Rayleigh-Ritz method to obtain the reduced eigenproblem as

$$\phi_i = X\tilde{\phi}_i \tag{31}$$

Modified form of equation (3) with shift is

$$K_m \phi_i = (\lambda_i - \mu) M \phi_i \tag{32}$$

Equation (32) can be rewritten as in the form

$$M(K_\mu^{-1} M)^\delta \phi_i = \frac{1}{(\lambda_i - \mu)^\delta} M \phi_i \tag{33}$$

Substituting equation (31) into equation (33) and multiplying the results by X^T, we can obtain the tridiagonalized standard eigenproblem of order $q << n$

$$T\tilde{\phi}_i = \frac{1}{(\lambda_i - \mu)^\delta} \tilde{\phi}_i \quad (i = 1, 2, \ldots, q) \tag{34}$$

where

$$T = X^T M (K_\mu^{-1} M)^\delta X = \begin{bmatrix} \alpha_1 \beta_1 & & & \\ \beta_1 \alpha_2 \beta_2 & & & \\ & \ddots & & \\ & & \alpha_{q-1} \beta_{q-1} & \\ & & \beta_{q-1} \alpha_q & \end{bmatrix} \tag{35}$$

Because T is tridiagonal, QR iteration is highly efficient in the calculation of eigenvalues of equation (34) (Bathe, 1996). Once eigenvalues are calculated, eigenvectors can be calculated by simple inverse iteration with shift equal to the corresponding eigenvalues. Two steps of inverse iteration are sufficient (Bathe, 1996).

2.3 Loss of Orthogonality

The Lanczos algorithm, involving orthogonalization with only the two preceding vectors at each step, is subjected to loss of orthogonality with respect to earlier vectors due to round-off errors. If such errors are not corrected when they reach a critical size, the Lanczos vectors may become linearly dependent. A remedy to prevent the loss of orthogonality is to use the Gram-Schmidt orthogonalization process at each step. In some cases, selective reorthogonalization may be sufficient. However, the Gram-Schmidt process is also sensitive to round-off errors, and it is actually necessary to perform the orthogonalization on all the preceding Lanczos vectors (Bathe, 1996). Such reorthogonalization scheme is called full reorthogonalization (Hughes, 1987). In this paper, full reorthogonalization process is used to retain the orthogonality of the Lanczos vectors as

$$\tilde{x}_i = \tilde{x}_i - \sum_{k=1}^{i} (\tilde{x}_i^T M x_k) x_k \tag{36}$$

2.4 Operation Count and Summary of Algorithm

Consider the number of Central Processor operations in order to obtain an estimate of the cost required for solving an eigenvalue problem. The actual cost must include, of course, the cost of the Peripheral Processor time. This time is, however, not considered in this investigation since it depends on the system and the programming technique. Let one operation equal to one multiplication which is nearly always followed by an addition. The steps for the matrix-powered Lanczos method with the operations are summarized in Table 1.

In the matrix-powered Lanczos algorithm, $\bar{x}_i$ is not calculated by direct δth power of the dynamic matrix $K_\mu^{-1}M$. It is obtained by δ-time forward-backward substitution. The algorithm with δ=1 corresponds to the conventional Lanczos algorithm. As δ increases in the matrix-powered Lanczos algorithm, the number of operations of forward-backward substitution increases. However, less operations of reorthogonalization process can compensate for the cost of forward-backward substitution. Generally, the cost of full reorthogonalization dominates the cost of Lanczos algorithm when the number of Lanczos step is large (Hughes, 1987). Because the power of dynamic matrix can reduce Lanczos steps, the cost of reorthogonalization can be reduced. For the eigenvalues of reduced system, QR itera-

Table 1. Operation Count for Matrix-Powered Lanczos Method

Operation	Calculation	Number of operations
Factorization	$K_\mu = LDL^T$	$(1/2)nm^2 + (3/2)nm$
Iteration $i = 1, 2, 3, \cdots q$		
Forward-backward substitution	$\bar{x}_i = (K_\mu^{-1}M)^\delta x_i$	$\delta\{n(2m+1)+2nm\}$
Multiplication	$\alpha_i = x_i^T M\bar{x}_i$	$n(2m+1)+n$
Multiplication	$\tilde{x}_i = \bar{x}_i - \alpha_i x_i - \beta_{i-1}x_{i-1}$	$2n$
Full reorthogonalization	$\tilde{x}_i = \bar{x}_i - \sum_{k=1}^{i}(\tilde{x}_i^T M x_k)x_k$	$i\{n(2m+1)+n+n\}$
Multiplication	$\beta_i = (\tilde{x}_i^T M\tilde{x}_k)^{1/2}$	$n(2m+1)+n+1$
Division	$x_{i+1} = \tilde{x}_i/\beta_i$	n
Repeat		
Solution of reduced system	$T\tilde{\phi}_i = (1/(\lambda_i-\mu)^\delta)\tilde{\phi}_i$	$\sum_{j=2}^{d} 6js_j + 10q^2$

Total operation

$$(1/2)nm^2 + (q^2+4q\delta+5q+3/2)nm + \{(3/2)q^2 + q\delta + (17/2)q\}n + 10q^2 + q + \sum_{j=2}^{q} 6js_j$$

Note: n=order of M and K
m=half-bandwidth of M and K
q=the number of calculated Lanczos vectors or order of T
s_j=the number of iterations of jth step in QR iteration

tion is effectively used because T is tridiagonal. In QR iteration, submatrix of order j is diagonalized by iteration in jth step (j=2, 3, $\cdots$ q). Each iteration requires $6j$ operations in jth step (Wilkinson, 1965). Therefore, the number of total operations will be $\sum_{j=2}^{q} 6js_j$ if the number of iterations in jth step is s_j. Eigenvectors of reduced system can be calculated by simple inverse iteration with shift equal to the corresponding eigenvalues. Two steps of inverse iteration are sufficient and the number of operations is $10q^2$ (Bathe, 1996).

3. Numerical Examples

A simple spring-mass system (Chen, 1993), a plane framed structure (Bathe and Wilson, 1972), a three-dimensional frame structure (Bathe and Wilson, 1972) and a three-dimensional building frame (Kim and Lee, 1999) are analyzed to verify the effectiveness of the matrix-powered Lanczos method. With the predetermined error norm of 10^{-6}, the structures are analyzed by two methods: the conventional Lanczos method and the matrix-powered Lanczos method, where the error norm is computed by the following equation

$$\varepsilon_i = \frac{\|K\phi_i - \lambda_i M\phi_i\|_2}{\|K\phi_i\|_2} \quad (37)$$

The number of operations and the required number of Lanczos vectors for calculating desired eigenpairs are compared. Maximum number of desired eigenpairs is set to be about ten percent of degree of freedom in each example. To examine the suitable power of the dynamic matrix, numerical examples are analyzed with varying the power of the dynamic matrix.

3.1 Simple Spring-Mass System

The first example is a simple spring-mass system. The finite element discretization of the system results in a diagonal mass matrix, a tridiagonal stiffness matrices of the following forms

$$M = mI \quad (38)$$

$$K = k\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \quad (39)$$

A system with order 100 is used in analysis. m and k are 1. Some results are shown in Table 2, Table 3 and Fig. 1. Table 2 represents the required number of Lanczos vectors for calculating desired eigenpairs. Table 3 summarizes the operation count for calculating desired eigenpairs and Fig. 1 compares the operation count graphically. In the tables and the figure, δ represents the power value of the dynamic matrix. The 1st power ($\delta = 1$) corresponds to the conventional Lanczos method. Table 2 shows that the convergence of the matrix-powered Lanczos method is better than that of the conventional Lanczos method. Table 3 and Fig. 1 show that the operation count of the matrix-powered Lanczos method is less than that of the conventional Lanczos method. In the calculation of 10th eigenpair, the 4th power ($\delta = 4$) gives failure in convergence. In some cases, high matrix power causes numerical instability (Zill and Cullen, 1992). The failure in convergence is due to the numerical instability.

Table 2. Required Number of Lanczos Vectors of Simple Spring-Mass System

No. of eigenpairs	$\delta = 1$ (ratio)	$\delta = 2$ (ratio)	$\delta = 3$ (ratio)	$\delta = 4$ (ratio)
2	9 (1.00)	7 (0.78)	6 (0.67)	5 (0.56)
4	14 (1.00)	11 (0.79)	9 (0.64)	8 (0.57)
6	18 (1.00)	14 (0.78)	12 (0.67)	11 (0.61)
8	21 (1.00)	17 (0.81)	15 (0.71)	14 (0.67)
10	25 (1.00)	20 (0.80)	18 (0.72)	*

*: Failure in convergence

3.2 Plane Framed Structure

The second example is a plane framed structure. The geometric configuration and the material properties are shown in Fig. 2. The structure was discretized using 210 beam elements resulting in system of dynamic equations with a total of 330 degrees of freedom. The half-bandwidth of mass and stiffness matrix is 32.

Some results are shown in Table 4, Table 5 and Fig. 3. It can be seen that results in convergence and operation count are similar to those of the previous example. In the calculation of 24th and 30th eigenpair, the 4th power ($\delta = 4$) gives failure in convergence.

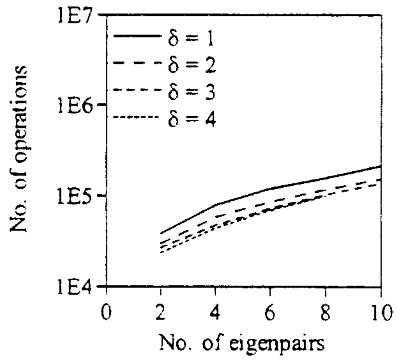
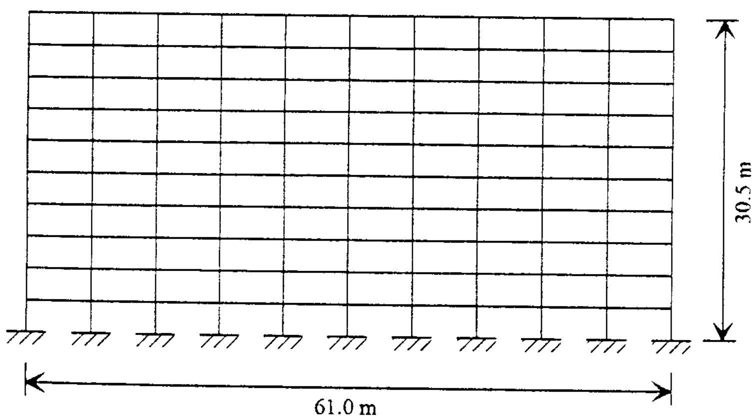


Fig. 1. Number of Operations of Simple Spring-Mass System

Table 3. Number of Operations of Simple Spring-Mass System

No. of eigenpairs	$\delta = 1$ (ratio)	$\delta = 2$ (ratio)	$\delta = 3$ (ratio)	$\delta = 4$ (ratio)
2	38663 (1.00)	29823 (0.77)	26954 (0.70)	23653 (0.61)
4	78922 (1.00)	58529 (0.74)	47567 (0.60)	44122 (0.56)
6	120458 (1.00)	85712 (0.71)	73040 (0.61)	69391 (0.58)
8	157649 (1.00)	117587 (0.75)	103055 (0.65)	99550 (0.63)
10	214729 (1.00)	154418 (0.72)	138122 (0.64)	*

*: Failure in convergence



$A = 0.2787\ m^2$, $I = 8.631 \times 10^{-3}\ m^4$, $E = 2.068 \times 10^7\ Pa$, $\rho = 5.154 \times 10^2\ kg/m^3$

Fig. 2. Plane Framed Structure

Table 4. Required Number of Lanczos Vectors of Plane Framed Structure

No. of eigenpairs	$\delta = 1$ (ratio)	$\delta = 2$ (ratio)	$\delta = 3$ (ratio)	$\delta = 4$ (ratio)
6	27 (1.00)	20 (0.74)	18 (0.67)	16 (0.59)
12	39 (1.00)	29 (0.74)	25 (0.64)	23 (0.59)
18	45 (1.00)	35 (0.78)	31 (0.69)	29 (0.64)
24	49 (1.00)	39 (0.80)	35 (0.71)	*
30	92 (1.00)	71 (0.77)	62 (0.67)	*

*: Failure in convergence

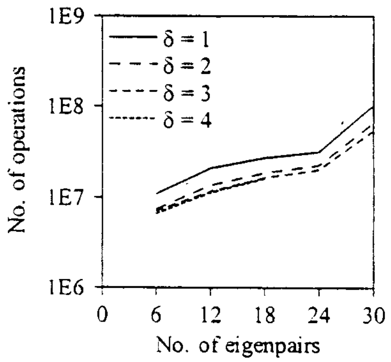


Fig. 3. Number of Operations of Plane Framed Structure

3.3 Three-Dimensional Frame Structure

The third example is a three-dimensional frame structure. The geometric configuration and the material properties are shown in Fig. 4. The structure was discretized using 100 beam elements resulting in system of dynamic equations with a total of 468 degrees of freedom. The half-bandwidth of mass and stiffness matrix is 137.

Some results are shown in Table 6, Table 7 and Fig. 5. Results in convergence and operation count are similar to those of the previous examples.

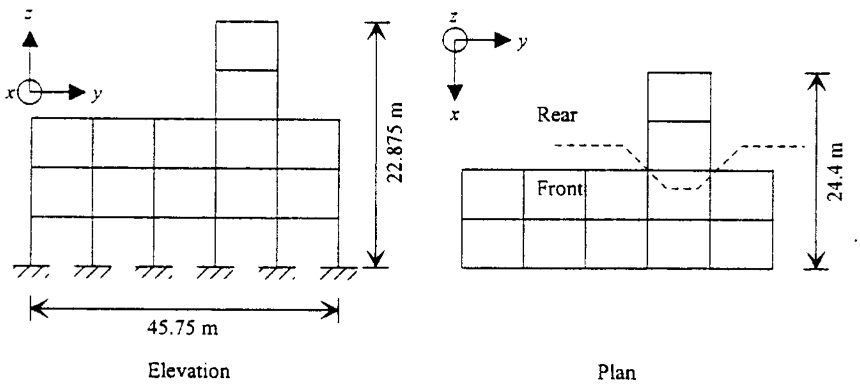
Table 6. Required Number of Lanczos Vectors of Three-Dimensional Frame Structure

No. of eigenpairs	$\delta = 1$ (ratio)	$\delta = 2$ (ratio)	$\delta = 3$ (ratio)	$\delta = 4$ (ratio)
10	28 (1.00)	21 (0.75)	19 (0.68)	17 (0.61)
20	48 (1.00)	37 (0.77)	34 (0.71)	31 (0.65)
30	64 (1.00)	51 (0.80)	46 (0.72)	43 (0.67)
40	98 (1.00)	77 (0.79)	68 (0.69)	64 (0.65)
50	121 (1.00)	94 (0.78)	84 (0.69)	78 (0.64)

3.4 Three-Dimensional Building Frame

The last example is a three-dimensional building frame. The geometric configuration and the material properties are shown in Fig. 6. The structure was discretized using 400 beam elements resulting in system of dynamic equations with a total of 1008 degrees of freedom. The half-bandwidth of mass and stiffness matrix is 149.

Some results are shown in Table 8, Table 9 and Fig. 7. As



Column in front building : $A = 0.2787\ m^2$, $I = 8.631\times10^{-3}\ m^4$
Column in Rear building : $A = 0.3716\ m^2$, $I = 10.789\times10^{-3}\ m^4$
All beams into x-Diretion : $A = 0.1858\ m^2$, $I = 6.473\times10^{-3}\ m^4$
All beams into y-Diretion : $A = 0.2787\ m^2$, $I = 8.631\times10^{-3}\ m^4$

$E = 2.068\times10^7\ Pa$
$\rho = 5.154\times10^2\ kg/m^3$

Fig. 4. Three-Dimensional Frame Structure

Table 5. Number of Operations of Plane Framed Structure

No. of eigenpairs	$\delta = 1$ (ratio)	$\delta = 2$ (ratio)	$\delta = 3$ (ratio)	$\delta = 4$ (ratio)
6	10908273 (1.00)	7429050 (0.68)	7072452 (0.65)	6633536 (0.61)
12	20855865 (1.00)	13578945 (0.65)	11688377 (0.56)	11237625 (0.54)
18	27029145 (1.00)	18676209 (0.69)	16508507 (0.61)	16047093 (0.59)
24	31581179 (1.00)	22516533 (0.71)	20164797 (0.64)	*
30	102944376 (1.00)	65994807 (0.64)	54112986 (0.53)	*

*: Failure in convergence

Table 7. Number of Operations of Three-Dimensional Frame Structure

No. of eigenpairs	$\delta = 1$ (ratio)	$\delta = 2$ (ratio)	$\delta = 3$ (ratio)	$\delta = 4$ (ratio)
10	71602154 (1.00)	50687925 (0.71)	48705515 (0.68)	46214349 (0.65)
20	181780512 (1.00)	124269611 (0.68)	116680070 (0.64)	108715163 (0.60)
30	307269560 (1.00)	215884077 (0.70)	192064376 (0.63)	182518601 (0.59)
40	684162222 (1.00)	453454527 (0.66)	378770940 (0.55)	356596304 (0.52)
50	1024104917 (1.00)	656188310 (0.64)	553972908 (0.54)	504420108 (0.49)

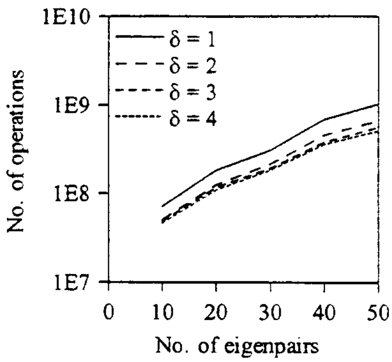
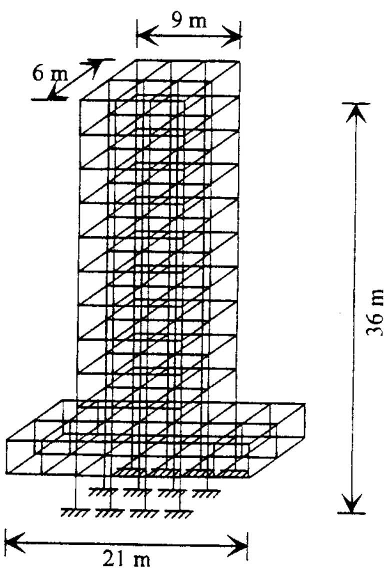


Fig. 5. Number of Operations of Three-Dimensional Frame Structure



$A = 0.01\ m^2, I = 8.3 \times 10^{-6}\ m^4$
$E = 2.1 \times 10^{11}\ Pa, \rho = 7850\ kg/m^3$

Fig. 6. Three-Dimensional Building Frame

Table 8. Required Number of Lanczos Vectors of Three-Dimensional Building Frame

No. of eigenpairs	$\delta = 1$ (ratio)	$\delta = 2$ (ratio)	$\delta = 3$ (ratio)	$\delta = 4$ (ratio)
20	46 (1.00)	36 (0.78)	*	*
40	84 (1.00)	66 (0.79)	*	*
60	137 (1.00)	108 (0.79)	*	*
80	145 (1.00)	122 (0.84)	*	*
100	148 (1.00)	148 (1.00)	*	*

*: Failure in convergence

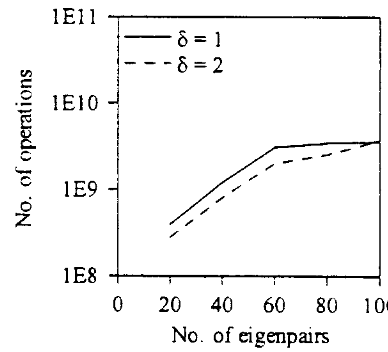


Fig. 7. Number of Operations of Three-Dimensional Building Frame

shown in the tables and the figure, results in convergence and operation count are similar to those of the previous examples. The 3rd and the 4th power ($\delta = 3, 4$) give failure in convergence.

The matrix-powered Lanczos method has less operation count than the conventional Lanczos method because the power of the dynamic matrix in the modified Lanczos recursion can improve the convergence. This fact is verified through above four numerical examples.

In the first and second examples, as the power value of

Table 9. Number of Operations of Three-Dimensional Building Frame

No. of eigenpairs	$\delta = 1$ (ratio)	$\delta = 2$ (ratio)	$\delta = 3$ (ratio)	$\delta = 4$ (ratio)
20	395079020 (1.00)	278717178 (0.71)	*	*
40	1196316954 (1.00)	801878160 (0.67)	*	*
60	3045578295 (1.00)	1993108128 (0.65)	*	*
80	3398746793 (1.00)	2509125474 (0.74)	*	*
100	3536190824 (1.00)	3625240574 (1.03)	*	*

* : Failure in convergence

dynamic matrix increases, the operation count decreases. However, the 4th power is numerically unstable. So, the 3rd power is the most suitable. In the third example, the 4th power gives the best solution. The 2nd power is the best in the last example. Considering four numerical examples, the 2nd power is suitable in the matrix-powered Lanczos method.

4. Conclusions

This paper extensively investigates the applicability of the Lanczos method using the power of the dynamic matrix to the eigenproblem of structures. The characteristics of the matrix-powered Lanczos method by the numerical results from examples are summarized as follows:

1. Since the power of the dynamic matrix can reduce the required number of Lanczos vectors, the matrix-powered Lanczos method has not only the better convergence but also the less operation count than the conventional Lanczos method.
2. The suitable power of the dynamic matrix that gives numerically stable solution in the matrix-powered Lanczos method is the second power.

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