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Algebraic Method for Sensitivity Analysis of Eigensystems with Repeated Eigenvalues

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Abstract

A simplified method for the computation of first-, second- and higher order derivatives of eigenvalues and eigenvectors associated with repeated eigenvalues is presented. Adjacent eigenvectors and orthonormal conditions are used to compose an algebraic equation. The algebraic equation developed can be used to compute derivatives of both eigenvalues and eigenvectors simultaneously. Since the coefficient matrix in the proposed algebraic equation is non-singular, symmetric and based on *N*-space, it is numerically stable and very efficient compared to previous methods. To verify the efficiency of the proposed method, the finite element model of the cantilever beam and a mechanical system in the case of a non-proportionally damped system are considered.

Keywords: *derivatives of eigenpairs, repeated eigenvalues, sensitivity analysis, second-order derivative of eigenpairs*

1. Introduction

Methods for computing the derivatives of eigenvalues and eigenvectors have been studied by many researchers in the past 30 years. The importance of obtaining sensitivities for eigenvalue problems stems from the fact that partial derivatives with respect to design parameters are extremely important for efficient design modifications under given situations, for gaining insight into the reasons of discrepancies between structural analyses and dynamic tests due to its design parameters change, and for indicating system model changes that will improve correlations between analyses and tests.

Most researches have focused on the use and computation of the first-order eigenpair sensitivities. The second- and higher order derivatives of eigenpairs are particularly important to predict the eigenpair of changed structures, which relies on the matrix Taylor series expansion. For large design parameter changes the linear approximation inherent in the use of first-order derivatives may be inadequate.

The sensitivity of eigenvalue problem with repeated eigenvalues has been a focus of recent interest. The most common circumstances under which repeated eigenvalues or nearly equal eigenvalues occur in typical structural or mechanical systems are instances where system symmetry exists, such as structures with two or more planes of reflective or cyclic symmetric or in the limiting case of axisymmetric bodies or certain reasons.

For the first-order derivatives of eigensystems with repeated eigenvalues, a generalization of Nelson's method (1976) was obtained by Ojalvo (1988) and modified by Mills-Curren (1988) and Dailey (1989). These methods are lengthy and complicated for finding derivatives of eigenvectors and clumsy for programming, because they basically follow Nelson's algorithm. Lee *et al.* (1999) developed an analytical method that gives exact solutions while it maintains *N*-space, but it finds eigenvalue derivative from classical method as before.

For the second- and higher derivatives of eigensystems with repeated eigenvalues, the method to calculate those has not been presented yet. For a real symmetric case with unrepeated eigenvalues, Brandon (1991) and Chen *et al.* (1994) calculated the second-order derivatives by writing the sensitivities as the series in the eigenvectors. Rudisill and Chu (1975) gave a direct method to calculate the sec-

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ond- and higher order derivatives of eigenvalues and eigenvectors. Jankovic (1994) calculated the second- and higher order derivatives of the general eigenproblem. Friswell (1994) extended Nelson's method for the calculation of the first-order derivatives of eigenvectors, or sensitivities, to the second- and higher order derivatives of eigenvectors.

In this paper, an efficient algebraic method for the first-, second- and higher order eigenpair sensitivities of damped systems with repeated eigenvalues is presented. The proposed method finds derivatives of eigenvalues and eigenvectors simultaneously from one equation. The proposed method does not use a state space equation (2N-space), instead of it, the method maintains N-space because a singularity problem is solved by using only one side condition. The algebraic equation of the proposed method may be efficiently solved by the LDL^T decomposition method. If the derivatives of the stiffness, mass and damping matrices can be analytically found, the proposed method can find the exact first, second- and higher order eigenpair derivatives.

2. Eigenpair Sensitivity in Damped Systems

2.1 First-order Derivatives

When an eigenvalue has multiplicity m and a design parameter is perturbed, the corresponding eigenvectors may split into as many as m distinct eigenvectors. Since the eigenspace spanned by eigenvectors corresponding to repeated eigenvalues is degenerate, any linear combination of eigenvectors can be an eigenvector. For the eigenvector derivative to be found, the adjacent eigenvectors which lie "adjacent" to the m (multiplicity of repeated eigenvalues) distinct eigenvectors appearing when a design parameter varies, must be calculated first. Otherwise, the eigenvectors would jump discontinuously with a varying design parameter. The derivatives of these adjacent eigenvectors are represented in this section.

The eigenvalue problem of a damped system can be expressed as

$$(\lambda^2\boldsymbol{M}+\lambda\boldsymbol{C}+\boldsymbol{K})\phi = \boldsymbol{0} \tag{1}$$

where M, C and K are the matrices of mass, damping and stiffness, respectively, and these are (n×n) symmetric matrices. M is positive definite and K is positive definite or semi-positive definite. The first step in finding derivatives of eigenvalues and eigenvectors of repeated eigenvalues is to find corresponding adjacent eigenvectors. Suppose that all eigenpairs are known and multiplicity of the eigenvalue λ_m is m. Define eigenvalue problem as:

$$\boldsymbol{M\Phi}_m\boldsymbol{\Lambda}_m^2+\boldsymbol{C\Phi}_m\boldsymbol{\Lambda}_m+\boldsymbol{K\Phi}_m = \boldsymbol{0} \tag{2}$$

where

$$\Lambda_m = \lambda_m I_m \text{ and } \Phi_m = [\phi_{i+1}\phi_{i+2}\cdots\phi_{i+m}] \tag{3}$$

where $\Phi_m(n\times m)$ is the matrix of eigenvectors corresponding to the repeated eigenvalues. $\boldsymbol{I}_m$ is the identity matrix of order m and λ_m is the eigenvalue of multiplicity m for the eigenspace spanned by the columns of Φ_m. The orthonormal condition for the (i+1)th eigenvector is as follows:

$$\boldsymbol{\phi}_{i+1}^T(2\lambda_{i+1}\boldsymbol{M}+\boldsymbol{C})\boldsymbol{\phi}_{i+1} = \boldsymbol{1} \tag{4}$$

Since the multiplicity is m, the orthonormal condition for the matrix Φ_m is as follows:

$$\boldsymbol{\Phi}_m^T(2\lambda_m\boldsymbol{M}+\boldsymbol{C})\boldsymbol{\Phi}_m = \boldsymbol{I}_m \tag{5}$$

Adjacent eigenvectors can be expressed in terms of Φ_m by an orthogonal transformation such as

$$\boldsymbol{X}_m = \boldsymbol{\Phi}_m\boldsymbol{T} \tag{6}$$

where $\boldsymbol{T}$ is an orthonormal transformation matrix and its order m;

$$\boldsymbol{T}^T\boldsymbol{T} = \boldsymbol{I}_m \tag{7}$$

The columns of the matrix $\boldsymbol{X}_m$ are the adjacent eigenvectors for which a derivative can be defined. It is natural that the adjacent eigenvectors also satisfy the orthonormal condition:

$$\boldsymbol{X}_m^T(2\lambda_m\boldsymbol{M}+\boldsymbol{C})\boldsymbol{X}_m = \boldsymbol{T}^T\boldsymbol{\Phi}_m^T(2\lambda_m\boldsymbol{M}+\boldsymbol{C})\boldsymbol{\Phi}_m\boldsymbol{T} = \boldsymbol{T}^T\boldsymbol{T} = \boldsymbol{I}_m \tag{8}$$

The next procedure is to find $\boldsymbol{T}$ and then to find $\boldsymbol{X}_m$. If design parameter α varies, the derivatives of the matrix Λ_m with respect to α can be expressed as

$$\Lambda_{m,\alpha} = \mathbf{diag}(\lambda_{i+1,\alpha}, \lambda_{i+2,\alpha}, \ldots, \lambda_{i+m,\alpha}) \tag{9}$$

where $(o)_{,\alpha}$ represents the derivative of (o) with respect to the design parameter α.

Consider following another eigenvalue problem to find $\boldsymbol{X}_m$.

$$\boldsymbol{MX}_m\boldsymbol{\Lambda}_m^2+\boldsymbol{CX}_m\boldsymbol{\Lambda}_m+\boldsymbol{KX}_m = \boldsymbol{0} \tag{10}$$

where the order of adjacent eigenvector matrix $\boldsymbol{X}_m$ is (n×m) and the order of eigenvalue matrix Λ_m is (m×m). Differentiating the above eigenvalue problem with respect to the design parameter α, and rearranging yields:

$$(\lambda_m^2 M + \lambda_m C + K)X_{m,\alpha} + (2\lambda_m M + C)X_m \Lambda_{m,\alpha} = -(\lambda_m^2 M_{,\alpha} + \lambda_m C_{,\alpha} + K_{,\alpha})X_m \tag{11}$$

Pre-multiplying at each side of equation (11) by Φ_m^T and substituting $X_m = \Phi_m T$ into it gives a new eigenvalue problem such as

$$DT = ET\Lambda_{m,\alpha} \tag{12}$$

where

$$D = \Phi_m^T(\lambda_m^2 M_{,\alpha} + \lambda_m C_{,\alpha} + K_{,\alpha})\Phi_m \tag{13}$$

and $E = -\Phi_m^T(2\lambda_m M + C)\Phi_m = -I_m$ (14)

The orthogonal transformation matrix T can be obtained by solving equation (12), and then the adjacent eigenvectors by relation $X_m = \Phi_m T$.

The proposed method starts with the equations of the derivative of the eigenvalue problem composed of the system matrices and the adjacent eigenvectors, equation (11), and the orthonormal condition, equation (8). Differentiating equation (8) with respect to the design parameter α gives

$$X_m^T(2\lambda_m M + C)X_{m,\alpha} + X_m^T M X_m \Lambda_{m,\alpha} = -0.5X_m^T(2\lambda_m M_{,\alpha} + C_{,a})X_m \tag{15}$$

Since the unknown or interested values are $X_{m,\alpha}$ and $\Lambda_{m,\alpha}$, equation (11) and equation (15) can be combined into a single matrix form as follows:

$$\begin{bmatrix} \lambda_m^2 M + \lambda_m C + K & (2\lambda_m M + C)X_m \\ X_m^T(2\lambda_m M + C) & X_m^T M X_m \end{bmatrix} \begin{Bmatrix} X_{m,\alpha} \\ \Lambda_{m,\alpha} \end{Bmatrix} = \begin{Bmatrix} -(\lambda_m^2 M_{,\alpha} + \lambda_m C_{,\alpha} + K_{,\alpha})X_m \\ -0.5X_m^T(2\lambda_m M_{,\alpha} + C_{,\alpha})X_m \end{Bmatrix} \tag{16}$$

where the order of coefficient matrix on the left side of equation (16) is $(n+m)\times(n+m)$ and the matrix of the right side of equation is $(n+m)\times m$. The first-order derivatives $X_{m,\alpha}$ and $\Lambda_{m,\alpha}$ can be found by solving equation (16).

2.2 Second- and Higher Order Derivatives

The second-order derivatives of eigenvalues and eigenvectors may be calculated by a procedure similar to that for the first-order derivatives. Differentiating the equation (11) of the derivative of the eigenvalue problem, with respect to a (possibly) different design parameter β gives

$$\begin{aligned}(\lambda_m^2 M + \lambda_m C + K)X_{m,\alpha\beta} + (2\lambda_m M + C)X_m \Lambda_{m,\alpha\beta} = \\ -(\tilde{F}_{m,\beta} + G_m \Lambda_{m,\beta})X_{m,\alpha} + (\tilde{F}_{m,\alpha} + G_m \Lambda_{m,\alpha})X_{m,\beta} \\ +(\tilde{\tilde{F}}_{m,\alpha\beta} + \tilde{G}_{m,\alpha}\Lambda_{m,\beta} + \tilde{G}_{m,\beta}\Lambda_{m,\alpha})X_m + 2\Lambda_{m,a}\Lambda_{m,\beta} M X_m\end{aligned} \tag{17}$$

Differentiating the equation of the derivative of the orthonormal condition, equation (15), with respect to a (possibly) different design parameter β gives

$$\begin{aligned}X_m^T(2\lambda_m M + C)X_{m,\alpha\beta} + X_m^T M X_m \Lambda_{m,\alpha\beta} = \\ -X_{m,\alpha}^T G_m X_{m,\beta} + X_m^T(\tilde{G}_{m,\beta} + 2M\Lambda_{m,\beta})X_{m,\alpha} \\ +X_m^T(\tilde{G}_{m,\alpha} + 2M\Lambda_{m,\alpha})X_{m,\beta} + 0.5X_m^T(\tilde{G}_{m,\alpha\beta} + 2M_{,\alpha}\Lambda_{m,\beta} \\ +2M_{,\beta}\Lambda_{m,a})X_m\end{aligned} \tag{18}$$

where

$$\begin{aligned}F_m &= [\lambda_m^2 M + \lambda_m C + K] \\ \tilde{F}_{m,\alpha} &= [\lambda_m^2 M_{,\alpha} + \lambda_m C_{,\alpha} + K_{,\alpha}] \\ \tilde{\tilde{F}}_{m,\alpha\beta} &= [\lambda_m^2 M_{,\alpha\beta} + \lambda_m C_{,\alpha\beta} + K_{,\alpha\beta}] \\ G_m &= [2\lambda_m M + C] \\ \tilde{G}_{m,\alpha} &= [2\lambda_m M + C_{,\alpha}] \\ \tilde{\tilde{G}}_{m,\alpha\beta} &= [2\lambda_m M_{,\alpha\beta} + C_{,\alpha\beta}]\end{aligned} \tag{19}$$

The unknown or interested values are $X_{m,\alpha\beta}$ and $\Lambda_{m,\alpha\beta}$ in equation (17) and equation (18) can be combined into a single matrix form as follows:

$$\begin{bmatrix} \lambda_m^2 M + \lambda_m C + K & (2\lambda_m M + C)X_m \\ X_m^T(2\lambda_m M + C) & X_m^T M X_m \end{bmatrix} \begin{Bmatrix} X_{m,\alpha\beta} \\ \Lambda_{m,\alpha\beta} \end{Bmatrix} = -\begin{Bmatrix} (\tilde{F}_{m,\beta} + G_m \Lambda_{m,\beta})X_{m,\alpha} + (\tilde{F}_{m,\alpha} + G_m \Lambda_{m,\alpha})X_{m,\beta} \\ +(\tilde{\tilde{F}}_{m,\alpha\beta} + \tilde{G}_{m,\alpha}\Lambda_{m,\beta} + \tilde{G}_{m,\beta}\Lambda_{m,\alpha})X_m \\ +2\Lambda_{m,\alpha}\Lambda_{m,\beta}^2 M X_m X_{m,a}^T \\ X_{m,\alpha}^T G_m X_{m,\beta} + X_m^T(\tilde{G}_{m,\beta}) + 2M\Lambda_{m,\beta})X_{m,\alpha} \\ +X_m^T(\tilde{G}_{m,\alpha} + 2M\Lambda_{m,\alpha})X_{m,\beta} \\ +0.5X_m^T(\tilde{\tilde{G}}_{m,\alpha\beta} + 2M_{,a}\Lambda_{m,\beta} + 2M_{,\beta}\Lambda_{m,\alpha})X_m \end{Bmatrix} \tag{20}$$

The second-order derivatives $X_{m,\alpha\beta}$ and $\Lambda_{m,\alpha\beta}$ can be found by solving equation (20). Note that equation (20) requires the first-order derivatives of the eigenvalues and eigenvectors obtained by equation (16), and the coefficient matrix of the left side is the same as that of equation (16).

If the second design parameter is equal to first design parameter $\alpha = \beta$, equation (20) can be simplified as follows:

$$\begin{bmatrix} \lambda_m^2 M + \lambda_m C + K & (2\lambda_m M + C)X_m \\ X_m^T(2\lambda_m M + C) & X_m^T M X_m \end{bmatrix} \begin{Bmatrix} X_{m,\alpha\alpha} \\ \Lambda_{m,\alpha\alpha} \end{Bmatrix}$$

$$= -\begin{Bmatrix} 2(\tilde{F}_{m,\alpha} + G_m \Lambda_{m,\alpha})X_{m,\alpha} + (\tilde{\tilde{F}}_{m,\alpha\alpha} + 2\tilde{G}_{m,\alpha}\Lambda_{m,\alpha} \\ +2M\Lambda_{m,\alpha}) \\ X_m X_{m;\alpha}^T G_m X_{m,\alpha} + X_m^T(2\tilde{G}_{m,\alpha} + 4M\Lambda_{m,\alpha}) \\ X_{m,\alpha} + 0.5X_m^T(\tilde{\tilde{G}}_{m,\alpha\alpha} + 4M_{,\alpha}\Lambda_{m,\alpha})X_m \end{Bmatrix} \quad (21)$$

The procedure for obtaining the second-order derivatives may be extended to third- and higher order derivatives. The k th-order derivatives are found by differentiating equation (10) and equation (8) k times with respect to the design parameters.

The first-, second- and higher order derivatives of the eigenvalues and eigenvectors can be obtained simultaneously from one augmented equation. It maintains N-space without use of the state space equation. The proposed method requires only corresponding eigenpair information different from modal methods, gives exact solutions and guarantees numerical stability. Numerical stability is proved in the next section.

3. Numerical Stability of the Proposed Method

Numerical stability is guaranteed by proving non-singularity of the coefficient matrix A^* in equation (16). To prove that the coefficient matrix A^* is non-singular, the determinant property is introduced as follows.

$$det(Y^T A^* Y) = det(Y^T)det(A^*)det(Y) \quad (22)$$

If $det(Y^T A^* Y) \neq 0$ is satisfied for an arbitrary non-singular matrix Y, then $det(A^*) \neq 0$ is also satisfied.

In this paper, the arbitrary non-singular matrix Y is assumed as

$$Y = \begin{bmatrix} \Psi & 0 \\ 0 & I_m \end{bmatrix} \quad (23)$$

where I_m is an identity matrix of order m and Ψ is a set of arbitrary independent vectors containing the adjacent eigenvectors of repeated eigenvalue λ_m of the systems, as follows

$$\Psi = [\phi_1 \phi_2 \Lambda \phi_{n-m} x_1 x_2 \Lambda x_m] \text{ when } X = [x_1 x_2 \ldots x_m] \quad (24)$$

where ψ's are arbitrary independent vectors chosen to be independent to the adjacent eigenvector x's. Since all the columns of the matrix Y are linear independent vectors, matrix Y is non-singular and it is invertible. Pre- and post-multiplying Y^T and Y to A^* yields:

$$Y^T A^* Y = \begin{bmatrix} \Psi & 0 \\ 0 & I_m \end{bmatrix}^T \begin{bmatrix} \lambda_m^2 M + \lambda_m C + K & (2\lambda_m M + C)X_m \\ X_m^T(2\lambda_m M + C) & X_m^T M X_m \end{bmatrix} \begin{bmatrix} \Psi & 0 \\ 0 & I_m \end{bmatrix}$$

$$= \begin{bmatrix} \Psi^T(\lambda_m^2 M + \lambda_m C + K)\Psi & \Psi^T(2\lambda_m M + C)X_m \\ X_m(2\lambda_m M + C)\Psi & X_m^T M X_m \end{bmatrix} \quad (25)$$

It is obvious that the last m columns and rows of the matrix $\Psi^T(\lambda_m^2 M + \lambda_m C + K)\Psi$ all have zero elements, which can be proved using equation (1) and $\Phi^T(\lambda_m^2 M + \lambda_m C + K)\Psi(\lambda_m^2 M + \lambda_m C + K)\Psi$ can be expressed as:

$$\Psi^T(\lambda_m^2 M + \lambda_m C + K)\Psi = \begin{bmatrix} \tilde{A} & 0 \\ 0 & 0 \end{bmatrix} \quad (26)$$

where $\tilde{A}$ is a non-zero $(n\text{-}m)\times(n\text{-}m)$ submatrix. The submatrix $\tilde{A}$ is a non-singular matrix having order of $(n\text{-}m)$ and rank of $(n\text{-}m)$, and it is given by eliminating the columns and rows having all zero elements from $\Psi^T(\lambda_m^2 M + \lambda_m C + K)\Psi$ of order n and rank $(n\text{-}m)$. That is, $det(\tilde{A}) \neq 0$.

By using the normalization condition, following equations are obtained.

$$\Psi^T(2\lambda_m M + C)X_m = \begin{bmatrix} \tilde{B} \\ I_m \end{bmatrix} \text{ and } X_m^T(2\lambda_m M + C)\Psi = \begin{bmatrix} \tilde{B} \\ I_m \end{bmatrix}^T \quad (27)$$

where $\tilde{B}$ is generally a non-zero rectangular matrix. Substituting equation (26) and (27), into equation (25) yields

$$Y^T A^* Y = \begin{bmatrix} \tilde{A} & 0 & \tilde{B} \\ 0 & 0 & I_m \\ \tilde{B}^T & I_m & X_m^T M X_m \end{bmatrix} \quad (28)$$

To find the determinant of the matrix, apply the determinant property of partitioned matrices such as

$$det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = detA \times det(D - CA^{-1}B) \quad (29)$$

Hence the determinant of equation (28) can be evaluated as

$$det(Y^T A^* Y)$$

$$= det(\tilde{A}) \times det\left(\begin{bmatrix} 0 & I_m \\ I_m & X_m^T M X_m \end{bmatrix} - \begin{bmatrix} 0 \\ \tilde{B}^T \end{bmatrix}[\tilde{A}]^{-1}[0 \ \tilde{B}]\right)$$

$$= -det(\tilde{A}) \neq 0 \quad (30)$$

The determinant of A^* is thus not equal to zero. In other words, the matrix A^* is non-singular. Hence, the proposed algorithm ensures the numerical stability.

4. Numerical Examples

To verify the efficiency of the proposed method, two examples with repeated eigenvalues are considered. The first example is a proportionally damped cantilever beam and the second example is a 5-DOF non-proportionally damped mechanical system.

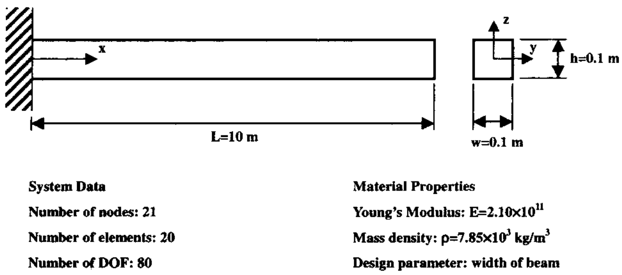

Fig. 1. Proportionally Damped Cantilever Beam

4.1 Proportionally Damped Cantilever Beam

As an illustrative example in case of the proportionally damped system with repeated eigenvalues, a cantilever beam with square cross section is considered as shown in Fig. 1. It is a FEM model composed of 20 elements and 21 nodes. Each node has four degrees of freedom. The structure has 80 degrees of freedom. Assume that the damping matrix is a linear combination of the stiffness and mass matrices as

$$C = \alpha K + \beta M \tag{31}$$

where α and β are the Rayleigh coefficients. In this example, α=0.0001 and β=0.0001 are used. The design parameter is taken as the beam width w.

The calculated lowest 12 eigenvalues and their derivatives of the cantilever beam are listed in the second and

Table 1. The Lowest 12 Eigenvalues of the Initial Cantilever Beam and Results of the Sensitivity Analysis

Mode number	Eigenvalues	First derivatives of eigenvalues	Second derivatives of eigenvalues
1, 2	−1.4279e−03 ± j5.2496e+00	−2.8057e−10 µj3.5347e−10	4.3916e−09 ±j1.0285e−08
3, 4	−1.4279e−03 ± j5.2496e+00	−2.2756e−02 ±j5.2494e+01	−2.7553e−01 µj6.1102e−02
5, 6	−5.4154e−02 ± j3.2895e+01	−6.6265e−10 ±j2.3445e−10	1.0084e−08 µj2.4918e−09
7, 8	−5.4154e−02 ±j3.2895e+01	−1.0818e+00 ±j3.2886e+02	−1.0806e+01 µj2.6913e+00
9, 10	−4.2409e−01 ±j9.2090e+01	6.9247e−10 µj6.9600e−10	−1.0391e−08 ±j1.1514e−08
11, 12	−4.2409e−01 ±j9.2090e+01	−8.4753e+00 ±j9.2029e+02	−8.4535e+01 µj1.8358e+01

Table 2. The Lowest 12 Eigenvalues and Approximated Eigenvalues of the Changed Cantilever Beam, Variations of Eigenpairs and Errors of Approximations

Mode number	Eigenvalues	Approximated eigenvalues	Variations of eigenpairs		Errors of approximations	
			Eigenvalues	Eigenvectors	Eigenvalues	Eigenvectors
1, 2	−1.4279e−03 ±j5.2496e+00	−1.4279e−03 ±j5.2496e+00	2.2281e−11	4.9628e−03	2.2283e−11	3.7376e−05
3, 4	−1.4556e−03 ±j5.3021e+00	−1.4555e−03 ±j5.3021e+00	1.0000e−02	9.9010e−03	2.6622e−08	1.0000e−04
5, 6	−5.4154e−02 ±j3.2895e+01	−5.4154e−02 ±j3.2895e+01	3.7084e−12	4.9628e−03	3.6899e−12	3.7376e−05
7, 8	−5.5241e−02 ±j3.3224e+01	−5.5236e−02 ±j3.3224e+01	9.9997e−04	9.9023e−03	1.6763e−07	1.0001e−04
9, 10	−4.2409e−01 ±j9.2090e+01	−4.2409e−01 ±j9.2090e+01	9.1400e−12	4.9628e−03	9.1432e−12	3.7376e−05
11, 12	−4.3261e−01 ±j9.3010e+01	−4.3256e−01 ±j9.3010e+01	9.9936e−03	9.9041e−03	4.6508e−07	1.0002e−04

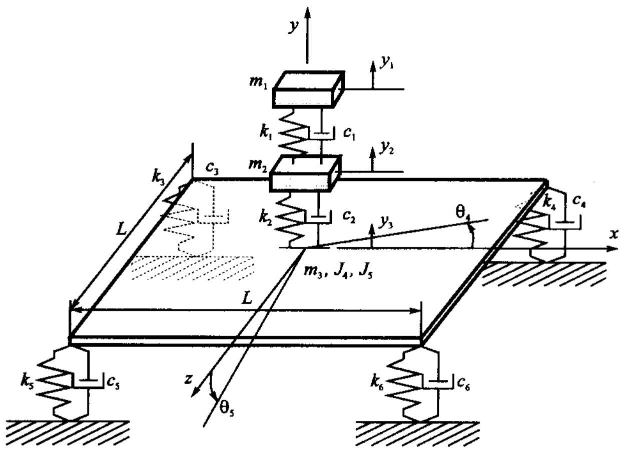

Fig. 2. 5-DOF Non-Proportionally Damped Mechanical System. m_1=200 kg, m_2=500 kg, m_3=1000 kg, k_1=10000 N/m, k_2=20000 N/m, k_3=k_4=k_5=k_6=1000 N/m, c_1=4 Ns/m, c_2=6 Ns/m, c_3=c_4=c_5=c_6=40 Ns/m

third columns of Table 1. The derivatives of the repeated eigenvalues are different in that one is close to zero while the other is not. Since the design parameter is the width of the beam, the repeated eigenvalues are split into distinct ones as the cross-section of the beam is no longer square after changing the width of the beam. The actual and approximate eigenvalues of the changed system with $\Delta w/w$=0.01 are represented in the second and third columns of Table 2. The next two columns in Table 2 are the variations of eigenvalues and eigenvectors between initial and changed ones and the last two are the errors of the approximations. The errors are reasonably smaller than the corresponding variations, and one can say that the proposed method gives good results for the case of repeated eigenvalues and for a proportionally damped system.

4.2 5-DOF Non-Proportionally Damped Mechanical System

As an illustrative example in case of the non-proportionally damped system with repeated eigenvalues, the 5-DOF mass, spring and damper system shown in Fig. 2 is considered. It is assumed that only vibrations in the vertical plane are possible. The design parameter is taken as the spring k_5.

Calculated derivatives of eigenvalues and eigenvectors are shown in Table 3 and 4. The derivatives of the repeated eigenvalues are different from each other because when k_5 is disturbed, the structural symmetry is broken and the repeated eigenvalues are split into distinct ones. The actual

Table 3 Eigenvalues of the Initial 5-DOF Mechanical System and Results of the Sensitivity Analysis

Mode number	Eigenvalues	First derivatives of eigenvalues	Second derivatives of eigenvalues
1, 2	−4.3262e−02 ±j1.5023e+00	9.6943e−07 ±j1.7995e−04	−1.4634e−08 µj2.4680e−07
3, 4	−2.4000e−01 ±j3.4558e+00	0.0000e+00 ±j0.0000e+00	0.0000e+00 ±j0.0000e+00
5, 6	−2.4000e−01 ±j3.4558e+00	0.0000e+00 ±j8.6811e−04	1.6409e−08 µj1.4913e−07
7, 8	−3.5202e−02 ±j6.1354e+00	−7.8926e−07 ±j2.9526e−05	−1.7067e−09 ±j1.4301e−08
9, 10	−2.4535e−02 ±j9.7000e+00	−1.8017e−07 ±j5.0001e−06	-6.8945e-11 ±j8.4803e-10

Table 4. Eigenvalues and Approximated Eigenvalues of the Changed 5-DOF Mechanical System, Variations of Eigenpairs and Errors of Approximations

Mode number	Eigenvalues	Approximated eigenvalues	Variations of eigenpairs		Errors of approximations	
			Eigenvalues	Eigenvectors	Eigenvalues	Eigenvectors
1, 2	−4.3243e−02 ±j1.5040e+00	−4.3253e−02 ±j1.5041e+00	1.1893e−03	4.6721e−03	8.1631e−07	2.9463e−05
3, 4	−2.4000e−01 ±j3.4558e+00	−2.4000e−01 ±j3.4558e+00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
5, 6	−2.4000e−01 ±j3.4645e+00	−2.4000e−01 ±j3.4645e+00	2.5039e−03	1.5461e−03	2.1632e−06	5.2014e−06
7, 8	−3.5210e−02 ±j6.1357e+00	−3.5210e−02 ±j6.1357e+00	4.8257e−05	1.0987e−03	1.1763e−07	2.5394e−06
9, 10	−2.4537e−02 ±j9.7000e+00	−2.4537e−02 ±j9.7000e+00	5.1624e−06	1.9422e−04	4.3893e−09	1.6332e−07

and approximate values of the changed system with $\Delta k_5/k_5$=0.01 are represented in the second and third columns of Table 4. The last four columns in Table 4 are variations of exact eigenpairs and the errors of approximations. Considering that the errors of the approximate eigenpairs are relatively smaller than the variations, the approximate eigenvalues and eigenvectors of the changed system are reasonable. Consequently, one can also say that the proposed method gives good results for the case of repeated eigenvalues and for a non-proportionally damped system.

5. Conclusions

A simple algorithm for the calculation of first- and second- eigenpair derivatives of the damped system with repeated eigenvalues is presented. The higher order derivatives can be calculated by a method similar to that for the second-order derivatives. The proposed method finds derivatives of eigenvalues and eigenvectors simultaneously from one augmented equation by solving a stable linear algebraic equation. This approach avoids the use of the state space equation and considers the damping problem explicitly by introducing a side condition of derivatives of normalization condition. Therefore, computation for the equation with N-order can be maintained and the errors in predicting the effect of structural approximation using first-order sensitivities can be reduced using the second- and higher order sensitivities without much increase in computing time.

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References

Bathe, K.J. (1982). *Finite Element Procedures in Engineering Analysis*, Englewood Cliffs, NJ: Prentice-Hall.

Brandon, J.A. (1991). "Second-Order Design Sensitivities to Assess the Applicability of Sensitivity Analysis." *AIAA Journal*, Vol. 29, pp. 135-139.

Chen, S.H., Song, D.T., and Ma, A.J. (1994). "Eigensolution Reanalysis of Modified Structures Using Perturbations and Rayleigh Quotients." *Communication in Numerical Methods in Engineering*, Vol. 10, pp. 111-119.

Chen, S.H., Qiu, Z., and Liu, Z. (1994). "Perturbation Method for Computing Eigenvalue Bounds in Structural Vibration Systems with Interval Parameters." *Communication in Numerical Methods in Engineering*, Vol. 10, pp. 121-134.

Dailey, R.L. (1989). "Eigenvector Derivatives with Repeated Eigenvalues." *AIAA Journal*, Vol. 27, pp. 486-491.

Friswell, M.I. (1994). "Calculation of Second and Higher Order Eigenvector Derivatives." *Journal of Guidance, Control, and Dynamics*, Vol. 18, pp. 919-921.

Jankovic, M.S. (1994). "Exact nth Derivatives of Eigenvalues and Eigenvectors." *Journal of Guidance, Control, and Dynamics*, Vol. 17, pp. 136-144.

Lee, I.W. and Jung, G.H. (1997). "An Efficient Algebraic Method for Computation of Natural Frequency and Mode Shape Sensitivities: Part II, Multiple Natural Frequencies." *Computers and Structures*, Vol. 62, pp. 437-443.

Lee, I.W., Kim, D.O., and Jung, G.H. (1999). "Natural Frequency and Mode Shape Sensitivities of Damped Systems: Part II, Multiple Natural Frequencies." *Journal of Sound and Vibration*, Vol. 223, pp. 413-424.

Mills-Curran, W.C. (1988). "Calculation of Derivatives for Structures with Repeated Eigenvalues." *AIAA Journal*, Vol. 26, pp. 867-871.

Nelson, R.B. (1976). "Simplified Calculations of Eigenvector Derivatives." *AIAA Journal*, Vol. 14, pp. 1201-1205.

Ojalvo, I.U. (1988). "Efficient Computation of Modal Sensitivities for Systems with Repeated Frequencies." *AIAA Journal*, Vol. 26, pp. 361-366.

Rudisill, C.S. and Chu, Y.Y. (1975). "Numerical Methods for Evaluating the Derivatives of Eigenvalues and Eigenvectors." *AIAA Journal*, Vol. 13, pp. 834-837.