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Modified Modal Methods for Calculating Eigenpair Sensitivity of Asymmetric Damped System

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Abstract

Many real systems such as moving vehicles on roads, missiles following trajectories and ships in sea water have asymmetric mass, damping and stiffness matrices. Eigen-sensitivity analysis methods for the symmetric damped system cannot be used in the asymmetric damped case. Therefore, a method for calculating eigenpair sensitivity of the asymmetric damped system is needed. To do this, a modal method employing a modal superposition idea was recently developed. Since the accuracy of the modal method is dependent on the number of modes used in calculation, the modal method needs higher eigenvectors to guarantee the accuracy. In large-scale systems, however, only a few lower modes are generally considered for the dynamic analysis. Hence, if the modal method is used to obtain the eigen-sensitivity of the large-scale system, the significant errors could not be avoided due to the lack of the information of higher modes. In this paper, the modified modal methods for computing the sensitivities of the eigenpairs of asymmetric damped system using a few lowest sets of modes are proposed. Numerical example shows that the proposed methods achieve better calculating efficiency than the previous modal method.
Keywords: *asymmetric, damped system, sensitivity*

1. Introduction

Natural frequencies and mode shapes of a structure are essential to understand the dynamic behavior of the structure. Design parameters, however, can be varied with damage, deterioration, corrosion, etc., resulting in the variation in natural frequencies and mode shapes. The variation of eigenpairs brings about the variation of the dynamic behavior of the system, which directly affects the stability of the structure. Therefore, the eigen-sensitivity analysis has played a central part in the structural stability analysis and has emerged as an important area of the research. Moreover, the eigenpair sensitivity is used in many areas, such as the optimization of a structure subject to natural frequencies, the system identification, the finite modeling updating and the structural control.

In one of the earliest works, Fox and Kapoor gave the exact expressions for the first derivative of eigenvalues and eigenvectors with respect to any design variable. Wang presented a modal method. This method approximates the eigenvector derivatives by a linear combination of the eigenvector. The modal methods employ a modal superposition idea. Therefore, the accuracy is dependent on the number of modes used in calculation. To guarantee the accuracy, the classical modal method needs higher eigenvectors. Recently, Zeng presented modified modal methods such as the multiple modal acceleration and multiple modal acceleration methods with shifted-poles for the complex eigenvectors in symmetric viscous damping systems. The modified modal methods achieved highly accurate results when only a few modes are used.

These eigenpair sensitivity methods are restricted to systems whose characteristic matrices are symmetric. However, in many dynamic problems, the inertia, stiffness and damping properties of the system cannot be represented by symmetric matrices. These kinds of problems typically arise in the dynamics of actively controlled structures and in many general non-conservative dynamic systems. For example, the behavior of structure in fluid, moving vehicles on roads, missile following trajectories, ship motion in sea water or the study of aircraft flutter. The asymmetric of damping and stiffness terms are often addressed in the con-

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text of gyroscopic and follower forces. It is too difficult to solve the eigenpair sensitivity of asymmetric systems by using the previous methods because of additional problems due to asymmetric characteristic matrices. To calculate the eigenpair derivatives in this case, many author have extended Fox and Kapoor's approach. Murthy and Haftka have written an excellent review on calculating the derivatives of eigenvalues and eigenvectors associated with general matrices. And Lee and Jung have obtained the derivaive of eigenvalues and eigenvectors using a first order formalism to reducing the calculating effort. Recently, Adhikari and Friswell proposed a modal method for asymmetric damped systems. This method used a modal superposition idea, too. Therefore, it also has disadvantages of aforementioned previous modal methods.

In this paper, by combining the modal method by Adhikari and Friswell and the modal acceleration with shifted-poles by Zeng, the modified modal methods for asymmetric damped system are presented. So the highly accurate modal methods for calculating the derivatives for asymmetric damped systems have been developed. In other words, the required number of eigenvectors could be smaller than the previous modal methods for the predetermined accuracy. A numerical example is presented to demonstrate the efficiency of the proposed modal methods.

2. Modal Methods for Asymmetric Damped Systems

The general equation of motion for an N-degree of freedom system with damping is

$$M\ddot{u}(t)+C\dot{u}(t)+Ku(t)=0 \tag{1}$$

where M, C and K are mass, damping and stiffness matrices, respectively. $u(t)$ is the vector of generalized coordinates, and t denotes time. The traditional restrictions of symmetry and positive definiteness are not imposed on M, C and K, however, it is assumed that M^{-1} exists.

Equation (1) can be rewritten in the following state-space form:

$$A\dot{x}(t)+Bx(t)=0 \tag{2}$$

where

$$A=\begin{bmatrix} C & M \\ M & 0 \end{bmatrix},\ B=\begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix},\ x(t)=\begin{Bmatrix} u(t) \\ \dot{u}(t) \end{Bmatrix} \tag{3}$$

From the equation (2), we obtain following two equations because of the asymmetricity of the system.

$$(sA+B)z=0 \tag{4}$$

$$y^T(sA+B)=0 \tag{5}$$

where s is the eigenvalue, z is the right eigenvector and y is the left eigenvector which is related to the right and left eigenvector of the second order system as

$$z=\begin{Bmatrix} u \\ su \end{Bmatrix},\ y=\begin{Bmatrix} v \\ sv \end{Bmatrix} \tag{6}$$

where u and v are the right and left eigenvector of the second order system.

For distinct eigenvalues, two normalizing conditions can be expressed [6] as follows:

$$y_j^T A z_j = 2s_j \tag{7}$$

$$\{u_j\}_{n_j}=\{v_j\}_{n_j} \tag{8}$$

where $\{*_j\}_{n_j}$ denotes the j-th element of a vector n_j is chosen so that the corresponding elements of the eigenvectors are as large as possible. Thus

$$|\{u_j\}_{n_j}||\{v_j\}_{n_j}| = \max_n |\{u_j\}_n||\{v_j\}_n| \tag{9}$$

To derive the eigenvalue derivatives, we differentiate the equation (4) with a design parameter α.

$$(s_{j,\alpha}A+s_jA_{,\alpha}+B_{,\alpha})z_j+(s_jA+B)z_{j,\alpha}=0 \tag{10}$$

Premultiplying y_j^T and using the equation (5), gives

$$s_{j,\alpha}=-\frac{y_j^T(s_jA_{,\alpha}+B_{,\alpha})z_{j,\alpha}}{y_j^TAz_j}=-\frac{y_j^T(s_jA_{,\alpha}+B_{,\alpha})z_j}{2s_j} \tag{11}$$

To derive the eigenvector derivatives, we can expand $z_{j,\alpha}$ and $y_{j,\alpha}$ as complex linear combinations of z_i and y_i, for all $i=1,\dots,2N$.

$$z_{j,\alpha}=\sum_{i=1}^{2N}a_{ji}z_i \tag{12}$$

$$y_{j,\alpha}=\sum_{i=1}^{2N}b_{ji}y_i \tag{13}$$

Substituting the equation (12) into equation (10) and premultiplying by y_j^T gives

$$y_k^T(s_{j,\alpha}A+s_jA_{,\alpha}+B_{,\alpha})z_j+\sum_{i=1}^{2N}a_{ji}y_k^T(s_jA+B)z_i=0 \tag{14}$$

Using the orthogonality relationship, we obtain

$$a_{jk}=-\frac{y_k^T(s_{j,\alpha}A+s_jA_{,\alpha}+B_{,\alpha})z_k}{2s_k(s_j-s_k)}\quad \forall k=1,\dots,2N;k\neq j \tag{15}$$

From similar procedure, we obtain

$$b_{jk} = -\frac{y_k^T(s_{j,\alpha}A + s_jA_{,\alpha} + B_{,\alpha})z_k}{2s_k(s_j - s_k)} \quad \forall k = 1,\ldots,2N; k \neq j \quad (16)$$

The expressions for a_{jk} and b_{jk} derived above are not valid when $k=j$. To obtain a_{jj} and b_{jj}, we begin by differentiating equation (7).

$$y_{j,\alpha}^T A z_j + y_j^T A_{,\alpha} z_j + y_j^T A z_{j,\alpha} = 0 \quad (17)$$

Substituting the equations (12) and (13) into the equation (17) and using the orthogonality relationship, gives

$$a_{jj} + b_{jj} = -\frac{y_j^T A_{,\alpha} z_j}{y_j^T A z_j} \quad (18)$$

The second equation is derived by using equation (8). If the n_j-th elements of the left and right eigenvectors remain equal then so do the corresponding elements of the derivatives. Thus

$$\{u_{j,\alpha}\}_{n_j} = \{v_{j,\alpha}\}_{n_j} = \{z_{j,\alpha}\}_{n_j} = \{y_{j,\alpha}\}_{n_j} \quad (19)$$

Substituting the equations (12) and (13) into equation (19), gives

$$b_{jj} - a_{jj} = \frac{1}{\{y_j\}_{n_j}} \sum_{\substack{k=1 \\ k \neq j}}^{2N} [a_{jk}\{z_k\}_{n_j} - b_{jk}\{y_k\}_{n_j}] \quad (20)$$

Therefore, the derivatives of eigenvectors are

$$z_{j,\alpha} = \left\{ \sum_{k=1,k\neq j}^{N} \left[\frac{z_k y_k^T}{2s_k(s_j - s_k)} + \frac{(z_k y_k^T)^*}{2s_k^*(s_j - s_k^*)} \right] + \frac{(z_j y_j^T)^*}{2s_j^*(s_j - s_j^*)} \right\} f_j + a_{jj} z_j \quad (21)$$

$$y_{j,\alpha} = \left\{ \sum_{k=1,k\neq j}^{N} \left[\frac{y_k z_k^T}{2s_k(s_j - s_k)} + \frac{(y_k z_k^T)^*}{2s_k^*(s_j - s_k^*)} \right] + \frac{(y_j z_j^T)^*}{2s_j^*(s_j - s_j^*)} \right\} f_j + b_{jj} y_j \quad (22)$$

where

$$f_j = -(s_{j,\alpha}A + sA_{,\alpha} + B_{,\alpha}) z_j \quad (23)$$

$$g_j = -(s_{j,\alpha}A + sA_{,\alpha} + B_{,\alpha}) y_j \quad (24)$$

3. Modified Modal Methods for Asymmetric Damped Systems

3.1 Modal Acceleration Method(MA)

The convergence of equations (21) and (22) is poor. To accurately calculate the eigenvector derivative, the higher modes are required. In a practical situation, there are only some lower modes available, and modal truncation errors will be significant. In response calculations, the modal acceleration approach is used to speed up the convergence and reduce the truncation errors.

Separate the response $z_{,\alpha}$ into a pseudostatic response z_{s0} and a dynamic correction response z_{d0} as follows:

$$z_{,\alpha} = z_{s0} + z_{d0} \quad (25)$$

where

$$z_{s0} = B^{-1} f \quad (26)$$

$$z_{d0} = z_{,\alpha} - z_{s0} \quad (27)$$

or

$$z_{d0} = (sA + B)^{-1} f - B^{-1} f \quad (28)$$

From the equation (7), we obtain

$$Y^T A Z = \begin{bmatrix} \ddots & & \\ & 2s_j^2 & \\ & & \ddots \end{bmatrix} \quad (29)$$

and

$$Y^T B Z = \begin{bmatrix} \ddots & & \\ & -2s_j^2 & \\ & & \ddots \end{bmatrix} \quad (30)$$

where Y and Z is the modal matrix to be formed by the right and left eigenvector, respectively.

Substituting the equations (29) and (30) into the equation (28), it yields

$$z_{d0} = Z \begin{bmatrix} \ddots & & \\ & \frac{1}{2s_k(s - s_k)}\left(\frac{s}{s_k}\right) & \\ & & \ddots \end{bmatrix} Y^T f \quad (31)$$

Hence, the total response is

$$z_{,\alpha} = B^{-1} f + \sum_{k=1}^{N} \left[\left(\frac{s}{s_k}\right) \frac{z_k y_k^T}{2s_k(s - s_k)} + \left(\frac{s}{s_k^*}\right) \frac{(z_k y_k^T)^*}{2s_k^*(s - s_k^*)} \right] f \quad (32)$$

Comparing Eq. (32) with Eq. (21), the derivative becomes

$$z_{j,\alpha} = \left\{ B^{-1} + \sum_{k=1,k\neq j}^{N} \left[\left(\frac{s_j}{s_k}\right) \frac{z_k y_k^T}{2s_k(s_j - s_k)} + \left(\frac{s_j}{s_k^*}\right) \frac{(z_k y_k^T)^*}{2s_k^*(s_j - s_k^*)} \right] + \left(\frac{s_j}{s_k^*}\right) \frac{(z_j y_j^T)^*}{2s_j^*(s_j - s_j^*)} \right\} f_j + a_{jj} z_j \quad (33)$$

By using the similar procedure, the left eigenvector derivative can be obtained as follows:

$$y_{,\alpha} = y_{s0} + y_{d0} \tag{34}$$

where

$$y_{s0} = B^{-T}g \tag{35}$$

$$y_{d0} = y_{,\alpha} - y_{s0} \tag{36}$$

or

$$y_{d0} = (sA+B)^{-T}g - B^{-T}g \tag{37}$$

Substituting the equations (29) and (30) into the previous equation, it yields

$$y_{d0} = Y\begin{bmatrix} \ddots & & \\ & \frac{1}{2s_k(s-s_k)}\left(\frac{s}{s_k}\right) & \\ & & \ddots \end{bmatrix} Z^T g \tag{38}$$

Hence, the total response is

$$y_{,a} = B^{-T}g + \sum_{k=1,k\neq j}^{N}\left[\left(\frac{s}{s_k}\right)\frac{y_k z_k^T}{2s_k(s-s_k)} + \left(\frac{s}{s_k}\right)\frac{(y_k z_k^T)^*}{2s_k^*(s-s_k^*)}\right] g \tag{39}$$

So the derivative becomes

$$y_{j,\alpha} = \left\{B^{-T} + \sum_{k=1,k\neq j}^{N}\left[\left(\frac{s_j}{s_k}\right)\frac{y_k z_k^T}{2s_k(s_j-s_k)} + \left(\frac{s_j}{s_k^*}\right)\frac{(y_k z_k^T)^*}{2s_j^*(s_j-s_k^*)}\right] + \left(\frac{s_j}{s_j^*}\right)\frac{(y_j z_j^T)^*}{2s_j^*(s_j-s_j^*)}\right\} g_k + a_{jj} y_j \tag{40}$$

When $k>j$, and $|s_j/s_k| < 1$, the convergence of MA is better than that of modal method. The effects of truncated higher modes could be reduced.

3.2 Multiple Modal Acceleration Method (MMA)

In the MA, the convergence of the series is speeded up through the preliminary calculation of the pseudostatic responses z_{s0} to the excitations $f = -(s_{,\alpha}A + sA_{,\alpha} + B_{,\alpha})z$. Based on the similar idea, if the "pseudostatic" response z_{s1} to the combination force of $f = -(s_{,\alpha}A + sA_{,\alpha} + B_{,\alpha})z$ and inertia force that comes from the response z_{s0} is preliminarily calculated, the convergence would be further improved for including the effects of the inertia. The pseudostatic response is

$$z_{,\alpha} = z_{s1} + z_{d1} \tag{41}$$

where

$$z_{s1} = B^{-1}f[I - sAB^{-1}]f \tag{42}$$

$$z_{d1} = z_{,\alpha} - z_{s1} \tag{43}$$

or

$$z_{d1} = (sA+B)^{-1}f - B^{-1}f[I - sAB^{-1}]f \tag{44}$$

Based on the equations (29) and (30), we have

$$z_{d1} = Z\begin{bmatrix} \ddots & & \\ & \frac{1}{2s_k(s-s_k)}\left(\frac{s}{s_k}\right)^2 & \\ & & \ddots \end{bmatrix} Y^T f \tag{45}$$

By the similar procedure as the modal acceleration approach, the right eigenvector derivative is given as

$$z_{j,\alpha} = \left\{B^{-1}(I - s_j AB^{-1}) + \sum_{k=1,k\neq j}^{N}\left[\left(\frac{s_j}{s_k}\right)^2\frac{z_k y_k^T}{2s_k(s_j-s_k)} + \left(\frac{s_j}{s_k^*}\right)^2\frac{(y_k z_k^T)^*}{2s_k^*(s_j-s_k^*)}\right] + \left(\frac{s_j}{s_j^*}\right)^2\frac{(z_j y_j^T)^*}{2s_j^*(s_j-s_j^*)}\right\} f_j + a_{jj} z_j \tag{46}$$

When $|s_j| < |s_k|$, there are

$$\left|\frac{s_j}{s_k}\right|^2 < \left|\frac{s_j}{s_k}\right|^1 < 1 \tag{47}$$

Therefore, the equation (46) converges faster than the equation (33). Based upon the similar procedure, when multiple modal accelerations are used, we will express the j-th right eigenvector derivative as

$$z_{j,\alpha} = \left\{B^{-1}\sum_{m=0}^{Ma-1}(-s_j AB^{-1})^m + \sum_{k=1,k\neq j}^{N}\left[\left(\frac{s_j}{s_k}\right)^{Ma}\frac{z_k y_k^T}{2s_k(s_j-s_k)} + \left(\frac{s_j}{s_k^*}\right)^{Ma}\frac{(z_k y_k^T)^*}{2s_k^*(s_j-s_k^*)}\right] + \left(\frac{s_j}{s_j^*}\right)^{Ma}\frac{(z_j y_j^T)^*}{2s_j^*(s_j-s_j^*)}\right\} f_j + a_{jj} z_j \tag{48}$$

By the similarly procedure, the left eigenvector derivatives is given as

$$y_{j,\alpha} = \left\{B^{-T}\sum_{m=0}^{Ma-1}(-s_j A^T B^{-1})^m + \sum_{k=1,k\neq j}^{N}\left[\left(\frac{s_j}{s_k}\right)^{Ma}\frac{y_k z_k^T}{2s_k(s_j-s_k)} + \left(\frac{s_j}{s_k^*}\right)^{Ma}\frac{(y_k z_k^T)^*}{2s_k^*(s_j-s_k^*)}\right] + \left(\frac{s_j}{s_j^*}\right)^{Ma}\frac{(y_j z_j^T)^*}{2s_j^*(s_j-s_j^*)}\right\} g_j + b_{jj} y_j \tag{49}$$

For a larger value of Ma, the second series in the equations (48) and (49) converges very fast. Only using several lower modes, the highly accurate results would be obtained.

3.3 Multiple Modal Accelerations with Shifted-Poles (MMA-SP)

There are two series in the equations (48): one is the first Ma terms of $(s_jA+B)^{-1}$ expanded by Taylor's series at the center of the s plane, and the other is the expansion of $(s_jA+B)^{-1}$ by complex eigenvectors. To speed up the convergence of the second series, the value of Ma should be as large as possible. In the first series, because

$$(-s_jAB^{-1})^m = Y^{-T}\begin{bmatrix} \ddots & & \\ & \left(\frac{s_j}{s_k}\right)^m & \\ & & \ddots \end{bmatrix} Y^T \quad (50)$$

The series is not convergent when $j>1$, $|s_j/s_1|>1$. The Ma couldn't be too large for the first series. Therefore, it is difficult to choose suitable Ma value for the two series. A key problem is that the expansion of $(s_jA+B)^{-1}$ at the center of the s plane is not convergent. We should expand $(s_jA+B)^{-1}$ in Taylor's series in the region of s_j. The term $(s_jA+B)^{-1}$ is expanded in Taylors series at the position β as

$$\begin{aligned}(s_jA+B)^{-1} &= [B+\beta A-(s_j-\beta)(-A)]^{-1} \\ &= (B+\beta A)^{-1}[I+(s_j-\beta)(B+\beta A)^{-1}A]^{-1} \\ &= (B+\beta A)^{-1}\sum_{m=0}^{Ma-1}[-(s_j-\beta)A(B+\beta A)^{-1}]^m \quad (51)\end{aligned}$$

Let Ma=1 and 2, respectively, through the similar previous procedure, and the j-th right eigenvector derivatives are formulated as

$$\begin{aligned}z_{j,\alpha} = \Bigg\{ &(B+\beta A)^{-1}\sum_{m=0}^{Ma-1}[-(s_j-\beta)A(B+\beta A)^{-1}]^m \\ &+\sum_{k=1,k\neq j}^{N}\left[\left(\frac{s_j-\beta}{s_k-\beta}\right)^{Ma}\frac{z_ky_k^T}{2s_k(s_j-s_k)}+\left(\frac{s_j-\beta}{s_k^*-\beta}\right)^{Ma}\right. \\ &\left.\frac{(z_ky_k^T)^*}{2s_k^*(s_j-s_k^*)}\right]+\left(\frac{s_j-\beta}{s_j^*-\beta}\right)^{Ma}\frac{(z_jy_j^T)^*}{2s_j^*(s_j-s_j^*)}\Bigg\}f_j+a_{jj}z_j \quad (52)\end{aligned}$$

By the similarly procedure, the left eigenvector derivatives is given as

$$\begin{aligned}y_{j,\alpha} = \Bigg\{ &\left\{\sum_{m=0}^{Ma-1}[-(s_j-\beta)(B+\beta A)^{-T}A^T]^m\right\}(B+\beta A)^{-T} \\ &+\sum_{k=1,k\neq j}^{N}\left[\left(\frac{s_j-\beta}{s_k-\beta}\right)^{Ma}\frac{y_kz_k^T}{2s_k(s_j-s_k)}+\left(\frac{s_j-\beta}{s_k^*-\beta}\right)^{Ma}\right. \\ &\left.\frac{(y_kz_k^T)^*}{2s_k^*(s_j-s_k^*)}\right]+\left(\frac{s_j-\beta}{s_j^*-\beta}\right)^{Ma}\frac{(y_jz_j^T)^*}{2s_j^*(s_j-s_j^*)}\Bigg\}g_j+b_{jj}y_j \quad (53)\end{aligned}$$

Let β satisfy

$$\left|\frac{s_j-\beta}{s_k-\beta}\right|<1,\quad k=1,2,\ldots,N\quad k\neq j \quad (54)$$

Since

$$\left|\frac{s_j-\beta}{s_k-\beta}\right|^{Ma}<\left|\frac{s_j}{s_k}\right|^{Ma}\ \text{were } k>j \quad (55)$$

the convergence of the second series is further speeded up. To calculate eigenvector derivative by the equations (52) and (53), thus, the accuracy will be increased, and the time consumed would be decreased.

4. Nmerical Example

A whirling beam whose system matrices are asymmetric is considered as a numerical example. This example is a gyroscopic system rotating with high speed and has a lumped mass in the center of the beam as shown in Fig. 1.

The equation of motion of a gyroscopic system is as follows:

$$M\ddot{u}(t)+(C+G)\dot{u}(t)+(K+H)u(t)=F(t) \quad (56)$$

where M, C, K and F are mass, damping, stiffness and external force matrices respectively, G is a gyroscopic matrix and H is a circulatory matrix that makes the system matrix asymmetric.

$$\begin{aligned}&M=\begin{bmatrix}M_{11} & 0\\ 0 & M_{22}\end{bmatrix}, G=\begin{bmatrix}0 & G_{12}\\ -G_{12} & 0\end{bmatrix}, C=\begin{bmatrix}C_{11} & 0\\ 0 & C_{22}\end{bmatrix} \\ &K=\begin{bmatrix}K_{11} & 0\\ 0 & K_{22}\end{bmatrix}, H=\begin{bmatrix}0 & H_{12}\\ -H_{12} & 0\end{bmatrix}\end{aligned} \quad (57)$$

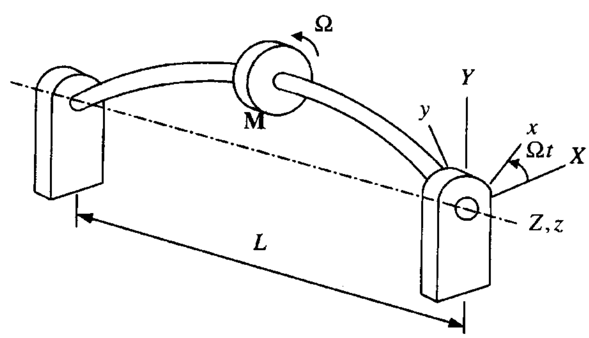

Fig. 1. The Whirling Beam

The element of which are given as

$$[M_{11}]_{ij} = [M_{22}]_{ij} = m_0 L\delta_{ij} + 2M\sin(i\pi/2)\sin(j\pi/2),\quad i,j = 1,2,\ldots,p$$

$$[G_{12}]_{ij} = -2\Omega[M_{11}]_{ij},\ [C_{11}]_{ij} = [C_{22}]_{ij} = (c+h)L\delta_{ij},\quad i,j = 1,2,\ldots,p$$

$$[K_{11}]_{ij} = 2(K_1 + K_2\cos(i\pi)\cos(j\pi))(i\pi/L)(j\pi/L) + EI_x(i\pi/L)^2(j\pi/L)^2 L\delta_{ij} - \Omega^2[M_{11}]_{ij},\quad i,j = 1,2,\ldots,p$$

$$[K_{22}]_{ij} = 2(K_1 + K_2\cos(i\pi)\cos(j\pi))(i\pi/L)(j\pi/L) + EI_y(i\pi/L)^2(j\pi/L)^2 L\delta_{ij} - \Omega^2[M_{11}]_{ij},\quad i,j = 1,2,\ldots,p$$

$$[H_{12}]_{ij} = -h\Omega L\delta_{ij},\quad i,j = 1,2,\ldots,p \quad (58)$$

and material properties are as follows:

$m_0 = 10\ kg/m, M = 10\ kg, L = 5\ m, EI_x = 4L^3/5\pi^2 Nm^2$, $EI_y = 9L^3/5\pi^2 Nm^2, K_1 = K_2 = L^2/20\ Nm, \Omega = \sqrt{21.6}$ $\pi\ rad\ s^{-1}, c = h = 1/4\ Nsm^{-1}$

The degrees of freedom of system are ten and the length of the beam L is used as a design parameter. The submatrices and have $M_{11}, M_{22}, G_{12}, C_{11}, C_{22}, K_{11}, K_{22}$ and H_{12} dimensions 5×5, the matrices *M*, *G*, *C*, *K* and H have dimensions 10×10 and the equations of motion and the eigenvalue problem have dimensions 20×20 when expressed in the state space form.

Table 1. Eigenvalues and Their Derivatives of System

Mode Number	Eigenvalues	Derivatives
1	−8.4987e−03 +2.3563e+00i	1.3251e−03 +1.5799e+00i
2	−2.7151e−03 +6.3523e+01i	2.2533e−03 +8.5934e−01i
3	1.6771e−02 +1.0548e+01i	3.3394e−03 +3.4034e−01i
4	6.3300e−02 +1.2534e+01i	8.8052e−03 +2.0616e-01i
5	2.3092e−01 +1.4079e+01i	2.1905e−02 +1.7729e−02i
6	−2.6645e−01 +1.5122e+01i	−2.3974e−02 −1.7731e−02i
7	−1.1330e−01 +1.6668e+01i	−8.8052e−3 −2.0608e−01i
8	−5.8579e-02 +1.8650e+01i	−3.7909e−03 -3.3918e−01i
9	−4.7285e−02 +2.2774e+01i	−2.2533e−03 −8.2215e−01i
10	−3.6890e−02 +2.6214e+01i	−1.2833e−03 −1.0644e+00i

Table 2. First right Eigenvector and Its Derivative

DOF Number	Eigenvector	Derivative
1	6.0874e−03 −6.2442e−06i	6.3118e−04 +6.3342e−06i
2	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
3	−7.4415e−03 +6.7358e−06i	−7.6005e−04 −7.1917e−06i
4	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
5	−2.8849e−02 −2.1839e−05i	−6.0508e−03 −1.3386e−05i
6	−1.2500e−05 +1.2110e−02i	1.0624e−05 −4.8858e−03i
7	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
8	+1.4785e−05 −1.4677e−02i	−1.2799e−05 +5.9162e−03i
9	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.00005e+00i
10	8.3733e−05 −5.7187e−02i	−3.7941e−05 +1.6957e−02i

Tables 1 to 3 show the eigenvalues and the associated eigenvectors and their derivatives with respect to a design variable.

To demonstrate the effectiveness of modified modal methods, the right eigenvector derivative is calculated using six modes. Table 4 shows the resulting errors, which defines

$$\varepsilon_j^i = \left|\frac{(z_{j,\alpha}^i)_{full} - (z_{j,\alpha}^i)_{few}}{(z_{j,\alpha}^i)_{full}}\right| \times 100(\%) \quad (5a)$$

where $(z_{j,\alpha}^i)_{full}$ is an *i*-th element of accurate derivative of the *j*-th right eigenvector by computing using full modes and $(z_{j,\alpha}^i)_{few}$ is an *i*-th element of derivative of the *j*-th right eigenvector by computing using a few lower modes. As seen from Table 4, the maximum errors of the MA, MMA and MMA-SP methods using six lower modes are 38.258%, 1.506% and 0.478%, respectively. The MMA-SP method gives the most exact results. It is verified that the method converges faster than the other methods.

Table 5 shows the resulting errors of MMA-SP method using 2, 4 and 6 lower modes. The maximum errors by using 2, 4 and 6 lower modes are 0.428%, 3.406% and

Table 3. First Left Eigenvector and Its Derivative

DOF Number	Eigenvector	Derivative
1	−6.0874e−03 +6.2442e−06i	−2.2630e−03 +1.8636e−05i
2	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
3	7.4415e−03 −6.7358e−06i	2.6937e−04 −2.3495e−05i
4	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
5	2.8849e−02 +2.1839e−05i	7.0186e−03 −1.0403e−04i
6	−1.2500e−05 +1.2110e−02i	−3.5515e−05 −5.3611e−03i
7	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
8	1.4785e−05 −1.4677e−02i	4.3471e−05 +6.4573e−03i
9	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
10	8.3733e−05 −5.7187e−02i	1.8429e−04 +1.3508e−02i

Table 4. Errors of Modified Modal Methods Using Six Modes (%) (MMA: Ma=2, MMA-SP: β=eigenvalue−1, Ma=2)

DOF Number	MA	MMA	MMA-SP
1	0.831	0.202	0.072
2	0.000	0.000	0.000
3	38.258	1.506	0.478
4	0.000	0.000	0.000
5	4.631	0.121	0.035
6	0.080	0.053	0.012
7	0.000	0.000	0.000
8	1.678	0.588	0.119
9	0.000	0.000	0.000
10	0.520	0.157	0.030

3.139%, respectively. As seen from the table 5, the maximum error of the method is less than 4% by using only two lower modes of ten modes.

5. Conclusions

The modified modal methods for the eigenpair derivatives of asymmetric damped systems have been derived in this paper. To obtain the accurate results, the previous modal method is needed all the eigenvalues and eigenvectors of the system. However, in practical case, only a few lower modes are available, resulting in the significant truncation errors in the previous method. On the other hand, the proposed modal methods, such as MA, MMA and MMA-SP, can calculate the derivatives of eigenvectors using only a few lower modes. By analyzing the numerical example, the effectiveness of the proposed methods has been verified.

Table 5. Errors of MMA-SP Method Using 2, 4 and 6 Lower Modes (%) (β=eigenvalue−1, Ma=2)

DOF Number	6 modes	4 modes	2 modes
1	0.072	3.406	2.089
2	0.000	0.000	0.000
3	0.478	0.454	3.139
4	0.000	0.000	0.000
5	0.035	0.035	0.052
6	0.012	0.626	0.383
7	0.000	0.000	0.000
8	0.119	0.114	0.542
9	0.000	0.000	0.000
10	0.030	0.030	0.039

The proposed modal methods are assumed that the system only has the distinct eigenvalues. To apply the proposed methods to the more practical problem, a study on the extension of the method to the system with repeated eigenvalues is required, and now in progress.

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