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Efficient Mode Superposition Methods for Non-classically Damped System

Sang-Won Cho*, Hung-Jo Jung[1] and In-Won Lee

Department of Civil and Environmental Engineering, Korea Advanced Institute of Science and Technology, 373-1 Kuseong-dong, Yuseong-gu, Daejeon 305-701, Korea

[1]Department of Civil and Environmental Engineering, Sejong University98, Gunja-dong, Gwangjin-Gu, Seoul 143-747, Korea

Abstract

The mode acceleration and the modal truncation augmentation methods improve the standard mode superposition method by complementing the portion of the truncated high modes. These methods have been used only in classically damped system. To use these methods with non-classical damping, the system has been approximated to the classically damped system. However, in this paper, each method is newly expanded to non-classically damped system for the efficient and accurate analysis without the approximations of the non-classically damped system. The state space approach is used for the expansion of each method. The applicability of expansion is verified analytically and numerically. The expanded modal truncation augmentation method is conditionally stable depending on the external loading whereas the expanded mode acceleration method is unconditionally stable. The stability condition of the expanded modal truncation augmentation method is suggested. In the stable case, both the methods give the same results. Two numerical examples are given to show characteristics of suggested methods.

Keywords: Mode superposition method, Non-classical damping, Modal displacement method, Mode acceleration method, Modal truncation augmentation method

1. Introduction

The methods of dynamic analysis are generally divided into two branches. One is the direct integration method and the other is the mode superposition method. The direct integration method is generally used for the analysis of the short duration loading such as an impulse whereas the mode superposition method is more efficient for the analysis of the long duration loading such as an earthquake. Unfortunately, the reduction process of the mode superposition method excludes the high mode component and alters the modal representation of loading, which can adversely affect the accuracy of calculated responses. To avoid errors from truncated high modes, the mode acceleration method (MA method) and the modal truncation augmentation method (MT method) improve the standard mode superposition method by complementing the truncated high modes. MA method was originally proposed by Williams (1987) and MT method was suggested by Dickens and Wilson (1980). Dickens *et al.* (1997) compared both the methods.

In the structural dynamics, it is common to assume that a system is classically damped. So, in practice, non-classically damped systems have been approximated to classically damped systems by neglecting the coupled damping terms (Warburton and Soni, 1987). In result, MA and MT methods were developed only for the classically damped systems. However, there are some important engineering problems where damping is no longer of the classical type and/or cannot be treated as such (Clough and Penzien, 1993). Amongst these are structures with base isolation (Kelly, 1993, 1994) and those with special damping devices (Inaudi *et al.* 1994). Another large class is that of the soil-structure interaction system, where the effect of non-classical damping is so important that it should be included in the analysis. In all those cases, the modes of vibration for the damped system no longer coincide with those of classically damped systems and in fact are complex as are the corresponding frequencies. The calculation of complex frequencies and modes of vibration has been considered in the literature (Hurty and Rubinstein, 1964 and Veletsos and Ventura, 1986) for discrete structures. Also, Oliveto and Santini (1996) show complex modal analysis of the dynamical response of continuous systems.

In this paper, MA and MT methods are newly expanded to non-classically damped system. As a result, the non-classical damping can be included in the analysis

*Corresponding author
Tel: +82-42-869-3658; Fax: +82-42-864-3658
E-mail: wood@kaist.ac.kr

without approximations. Besides, MA and MT methods give the results more efficiently than the standard mode superposition method.

The state space approach is used to develop each method. Such a technique (Newland, 1989 and Igusa *et al.*, 1984) is a general method of dynamic analysis of multi-degree-of-freedom dynamic systems, which is applicable to both the classically and non-classically damped systems without neglecting the coupled damping. The applicability of suggested methods is verified by closed form solutions and numerical examples.

The first section of this paper gives an overview of the standard mode superposition method for classically damped systems and non-classically damped systems. The second and third sections describe newly expanded MA and MT methods for the non-classically damped systems, respectively. In the fourth section, characteristics of expanded MT method is described and the stability condition for the expanded MT method is suggested. In the last section, the analytical conclusions are validated by two numerical examples.

2. Mode Superposition Method for Non-classically Damped Systems

Let the equations of motion of a linear structural system be given by:

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{C}\dot{\mathbf{u}}(t)+\mathbf{K}\mathbf{u}(t)=\mathbf{R}_0 r(t) \tag{1}$$

where **M**, **C** and **K** are an n by n mass, damping and stiffness matrices, respectively; and $\ddot{\mathbf{u}}(t)$, $\dot{\mathbf{u}}(t)$ and $\mathbf{u}(t)$ are an n by 1 acceleration, velocity and displacement vector, respectively. The applied loading is composed of two parts: $\mathbf{R}_0$ is the invariant spatial portion and $r(t)$ is the time varying portion.

For the modal response analysis, the physical coordinates of equation (1) are transformed to modal coordinates, $q(t)$, by a retained set of eigenvectors of the system, Φ:

$$\mathrm{u}(t)=\Phi q(t) \tag{2}$$

where Φ are determined from the general eigenvalue problem:

$$\mathbf{K}\Phi=\mathbf{M}\Phi\Omega^2 \tag{3}$$

where Ω^2 are eigenvalues corresponding to the eigenvectors: $\Phi=[\phi_1\ \phi_2\ \cdots\ \phi_n]$.

If the system is classically damped, equation (2) is used to transform equation (1) to n decoupled equations of modal coordinates:

$$\ddot{q}_i+2\beta_i\omega_i\dot{q}_i+\omega_i^2 q_i=\phi_i^T\mathbf{R}_0 r(t),\ i=1\ ,2,\ \cdots,\ n \tag{4}$$

where ϕ_i, ω_i and β_i are the i th eigenvector, natural frequency and damping ratio of the system, respectively.

The mode displacement (MD) method, the standard procedure for determining responses, consists of expanding the modal responses, solved from equation (4), into approximate physical responses using the retained m modes (Craig, 1981 and Meirovitch, 1967):

$$\mathbf{u}(t)=\sum_{i=1}^{m}\phi_i q_i(t)\quad (m<<n) \tag{5}$$

where n is the total degree of the freedom of system and m is the number of retained modes. MD method gives approximated responses because m, the number of retained modes, is much smaller than n, the total degree of the freedom of the system.

If the system is non-classically damped, equation (1) cannot be transformed to decoupled equations because the damping matrix, **C** cannot be diagonalized. To apply mode superposition method to non-classically damped system, the second-order differential equation (1) can be transformed into a first-order differential equation by doubling the size of the system, which is the state space approach (Newland, 1989 and Igusa *et al.*, 1984):

$$\mathbf{B}\dot{\mathbf{y}}(t)-\mathbf{A}\mathbf{y}(t)=\mathbf{F}_0 r(t) \tag{6}$$

where **A** and **B** is the $2n$ by $2n$ matrix composed of **M**, **C** and **K** and **y**(t) is the $2n$ by 1 vector:

$$\mathbf{B}=\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix},\ \mathrm{A}=\begin{bmatrix}-\mathbf{K} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{bmatrix},\ \mathbf{y}(t)=\begin{Bmatrix}\mathbf{u}(t)\\ \dot{\mathbf{u}}(t)\end{Bmatrix},\ \mathbf{F}_0=\begin{Bmatrix}\mathbf{R}_0\\ 0\end{Bmatrix} \tag{7}$$

where **0** is the n by 1 zero vector.

For a modal response analysis, the physical coordinates of equation (6) are transformed to modal coordinates, z(t), by a retained set of eigenvectors of the system, Ψ.

$$\mathbf{y}(t)=\Psi z(t) \tag{8}$$

where Ψ are determined from the eigenvalue problem:

$$\mathbf{A}\psi_i=s_i\mathbf{B}\psi_i \tag{9}$$

where ψ_i, i th eigenvector, and s_i, corresponding eigenvalue, may occur in complex-conjugate pairs, respectively.

For non-classically damped systems, equation (6) can be transformed to $2n$ decoupled modal equations as in equation (10) by using the modal transformation of equation (8):

$$\dot{z}_i-s_i z_i=\psi_i^T\mathbf{F}_0 r(t)\quad i=1\ ,2,\ \cdots,\ 2n \tag{10}$$

where modal coordinates, z_i, may be complex.

Physical responses can not be in complex. MD method for non-classically damped systems can give approximate physical responses using retained $2q$ modes of complex-conjugate pairs:

$$\mathbf{y}(t)=\sum_{i=1}^{2q}\psi_i z_i(t)\quad (q<<n) \tag{11}$$

where n is the degree of the freedom of original system and $2q$ is the number of retained modes.

3. Mode Acceleration Method for Non-classically Damped Systems

For the modal response analysis, the mode superposition method generally uses the lower $2q$ modes among the total $2n$ modes as in equation (11). The truncated high modes which are not retained in the modal response analysis, induce the errors (Warburton and Soni, 1987). MA and MT methods reduce these errors by complementing the truncated high modes.

But the limitations of non-classically damped systems, such as the difficulty of diagonalization and the inefficiency of generating the eigenvectors, restrict the use of these methods to the classically damped systems. To be employed in non-classically damped system, MA and MT methods are newly expanded to the non-classically damped systems by using the state space approach.

To develop MA algorithm, equation (4) is rewritten as

$$\mathbf{y}(t)=\mathbf{A}^{-1}\mathbf{B}\dot{\mathbf{y}}(t)-\mathbf{A}^{-1}\mathbf{F}(t) \tag{12}$$

Using equation (11), the displacements by expanded MA algorithm, $\mathbf{y}_{ma}(t)$, are given by:

$$\mathbf{y}_{ma}(t)=\mathbf{A}^{-1}\mathbf{B}\sum_{i=1}^{2q}\psi_i\dot{z}_i-\mathbf{A}^{-1}\mathbf{F}(t) \quad (q<<n) \tag{13}$$

The responses by expanded MA method consists of two parts as mentioned by Dickens and *et al.* (1997) in classically damped systems.

$$\mathbf{y}_{ma}(t)=\mathbf{y}_s(t)+\mathbf{y}_{tma}(t) \tag{14}$$

The first portion, $\mathbf{y}_s(t)$, is the displacement which is calculated from the retained modes. Thus $\mathbf{y}_s(t)$ is the same as $\mathbf{y}(t)$ given in equation(11) of MD method. The second part, $\mathbf{y}_{tma}(t)$, is the displacements which complements truncated high modes. $\mathbf{y}_{tma}(t)$ is calculated by subtracting equation (11) from equation (13):

$$\mathbf{y}_{tma}=-\mathbf{A}^{-1}\mathbf{R}_t\, r(t) \tag{15}$$

In the equation (15), $\mathbf{R}_t$ is force truncation vector defined as

$$\mathbf{R}_t=\mathbf{F}_0-\mathbf{R}_s \tag{16}$$

where $\mathbf{R}_s$ is the modally represented spatial load vector:

$$\mathbf{R}_s=\mathrm{B}[\psi_i\overline{\psi}_i][\psi_i\overline{\psi}_i]^T\,\mathbf{F}_0 \tag{17}$$

where ψ_i and $\overline{\psi}_i$ are eigenvectors in complex-conjugate pairs. Modal-physical representation of displacements and external loads such as $\mathbf{R}_s$ and $\mathbf{R}_t$ are well documented by J. M. Dickens *et al.* (1997) But in this paper, the row size of each vector, $\mathbf{R}_s$ and $\mathbf{R}_t$, is doubled to the $2n$ by 1 vector, because the state space approach is used. $\mathbf{R}_s$ for the non-classically damped system is calculated from the complex-conjugate pairs of eigenvector, because $\mathbf{R}_s$ has physical meaning of modally represented spatial load vector, which can not be in the complex domain.

4. Modal Truncation Augmentation Method for Non-classically Damped Systems

The responses by MT method consist of two parts in a similar way with MA method.

$$\mathbf{y}_{mt}(t)=\mathbf{y}_s(t)+\mathbf{y}_{tmt}(t) \tag{18}$$

The first portion, $\mathbf{y}_s(t)$, is the displacement given in equation (11), which is calculated from the retained modes. The second part, $\mathbf{y}_{tmt}(t)$, supplements the truncated high modes. MT method approximates the non-modally represented solution by MT vector, $\mathbf{P}$ which supplements the effect of truncated high modes. MT vector is determined by solving for the displacement vector, $\mathbf{P}$,due to the force truncation vector of equation (16):

$$\mathbf{A}\overline{\mathbf{P}}=\mathbf{R}_t \tag{19}$$

MT vector, P, can be formed

$$\mathbf{P}=\frac{1}{\alpha}\overline{\mathbf{P}} \tag{20}$$

where $\alpha=(\overline{\mathbf{P}}^T\mathbf{B}\overline{\mathbf{P}})^{1/2}$. MT vector has a mathematical consistence with Rayleigh-Ritz approximation where the assumed Ritz basis vectors are derived using the spatial force truncation vector. MT vector is orthogonal on the matrix **A** and **B** but does not satisfy the eigenvalue problem at each equation.

For a modal response analysis by MT method, a transformation is used as:

$$\mathbf{y}_{tma}=\mathbf{P}z_P(t) \tag{21}$$

Using this transformation to reduce equation (6):

$$\mathbf{P}^T\mathbf{B}\mathbf{P}\dot{z}_P(t)-\mathbf{P}^T\mathbf{A}\mathbf{P}z_P(t)=\mathbf{P}^T\mathbf{R}_t r(t) \tag{22}$$

Equation (22) is rewritten to be solved for $z_P(t)$:

$$\dot{z}_P(t)-s_P z_P(t)=\mathbf{P}^T\mathbf{R}_t r(t) \tag{23}$$

where $s_P=\mathbf{P}^T\mathbf{A}\mathbf{P}$. $z_P(t)$ is calculated from equation (23), and then $z_P(t)$ is back-transformed to yield MT solution which represents the non-retained modes by using equation (21).

Both MA and MT methods approximate the non-modally represented solution by compensating truncated high modes. In classically damped systems, it would be expected that MT method gives better results than MA method because of added dynamics (Dickens *et al.*, 1997). In the non-classically damped systems, however, MT method has different characteristics from those in the classically damped systems. The characteristics of expanded MT method are discussed in two points, the stability and characteristics of the stable solution.

5. The Characteristics of the Modal Truncation Augmentation Method for Non-classically Damped Systems

5.1. Stability

To check the stability of MT method, the general solution of equation (24) is solved for the initial condition, $z_P(0) = 0$, when the harmonic force, $r(t) = \sin(\bar{\omega} t)$, is applied:

$$z_P(t) = \frac{\mathbf{P}^T\mathbf{R}_t}{s_P^2+\bar{\omega}^2}(\bar{\omega}\, e^{s_P t} - s_P\sin(\bar{\omega} t - \bar{\omega}\cos(\bar{\omega} t)) \tag{24}$$

The applied force is assumed to be harmonic function because the external force like an earthquake can be represented by combination of harmonic functions.

Equation (24) is stable when the s_P satisfies the stability condition:

$$s_P = \mathbf{P}^T\mathbf{A}\mathbf{P} < 0 \tag{25}$$

If the condition of equation (25) is not satisfied, MT solution would be divergent in time. Therefore, MT method can be applicable to non-classically damped systems, which satisfies the stability condition of equation (25). Because $\mathbf{A}$ is generally indefinite (Ibrahimbegovic *et al.*, 1990), the stability of MT method depends on MT vector, $\mathbf{P}$, which means that stability is affected by the input load.

5.2. Characteristics of the solution by MT method

Once the stability condition is satisfied, the first term of equation (24) decays very rapidly. At the third term, $\bar{\omega}$ is the frequency of external loading and s_P is calculated from $\mathbf{A}$ and $\mathbf{P}$. In the case of $|s_P| >> \bar{\omega}$, equation (24) can be simplified:

$$z_P(t) = \frac{-\mathbf{P}^T\mathbf{R}_t}{s_P}\sin(\bar{\omega} t) \tag{26}$$

For the most real structures, $|s_P|$ is much bigger than $\bar{\omega}$, because the elements of $\mathbf{A}$ are material properties of structure whereas $\bar{\omega}$ is the frequency of external loading. Then $z_P(t)$ can be back-transformed to yield MT solution for the displacements which present the non-retained modes:

$$\mathbf{y}_{t_{mt}}(t) = \mathbf{P}z_P(t) = \frac{\mathbf{P}\mathbf{P}^T\mathbf{R}_t}{s_P}\sin(\bar{\omega} t) \tag{27}$$

Using the relation of equation (19) and (20), MT vector, $\mathbf{P}$, can be represented as follows:

$$\mathbf{P} = \frac{1}{\alpha}\mathbf{A}^{-1}\mathbf{R}_t \tag{28}$$

Substituting equation (28) into equation (27), the displacements by the MT solution is as follows.

$$\mathbf{y}_{t_{mt}} = \mathbf{A}^{-1}\mathbf{R}_t = \frac{\mathbf{P}^T\mathbf{R}_t}{\alpha s_P}\sin(\bar{\omega} t) \tag{29}$$

MT solution is similar to MA solution in equation (15). If the previous relations such as $\mathbf{R}_t = \alpha\mathbf{A}\mathbf{P}$ and $s_P = \mathbf{P}^T\mathbf{A}\mathbf{P}$ are used, it can be shown that MT solution is identical with MA solution:

$$\mathbf{y}_{t_{mt}} = -\frac{\mathbf{P}^T\hat{\mathbf{R}}_t}{\alpha s_P}\mathbf{A}^{-1}\hat{\mathbf{R}}_t = \sin(\bar{\omega} t) = -\mathbf{A}^{-1}\hat{\mathbf{R}}_t\sin(\bar{\omega} t) = \mathbf{y}_{t_{ma}} \tag{30}$$

In summary, the applicability of expanded MT method is limited by the stability condition in equation (25). When the stability condition is satisfied, MT method is stable and applicable to non-classically damped systems and it is proved that the result of MT solution is identical to MA solution as in equation (26)-(30).

6. Numerical Examples

To verify suggested methods, two numerical examples are presented. In the first example, the efficiency of each method is compared. In the second example, convergent and divergent cases of MT method are shown, respectively. In the stable case of MT method, it can be seen that the result is identical with MA method. Each result is normalized by the result of the direct integration method which is exact solution of the system.

6.1. Cantilever beam subjected to earthquake

This example shows that expanded MA and MT methods are more efficient than MD method as in the classically damped systems. The system in Fig. 1, is a cantilever beam with a lumped translational viscous-damper attached at each node. The cantilever beam is modeled by 10 identical beam elements. The Youngs modulus, area, inertia and density of beam are respectively 2.07×10^6 MPa, 26 cm^2, 0.52 cm^4 and 2.01 $\times 10^{-1}$ N/cm^3. The damping coefficient of the tangential damper is 0.18 N sec/cm. The load applied to the system is El-Centro earthquake. Table 1 shows the complex-conjugate pairs of eigenvalue.

The results are shown from Fig. 2 to Fig. 5. In the legend, md1 means that MD method uses the lowest one eigenvector. md2 denotes that MD method uses the lowest two eigenvectors for the modal response analysis.

For the bending moment at each node, MA and MT solutions are shown in Fig. 2 and 3. In these figures, MA and MT methods are more efficient than MD method

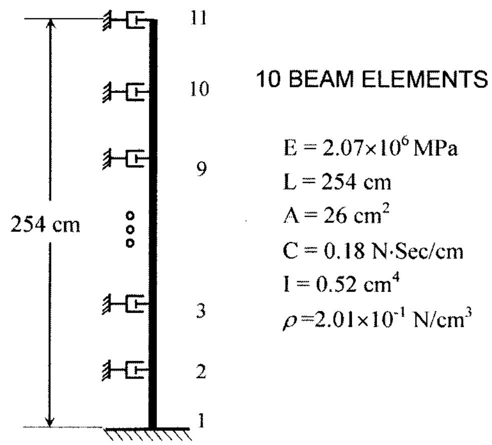

Figure 1. Beam configuration.

Table 1. Eigenvalue set

Mode number	Eigenvalues
1	−4.43482 − 39.29620i
2	−4.43482 + 39.29620i
3	−88.4454 − 231.3995i
4	−88.4454 + 231.3995i
5	−677.3535 − 147.892i
6	−677.3535 + 147.892i

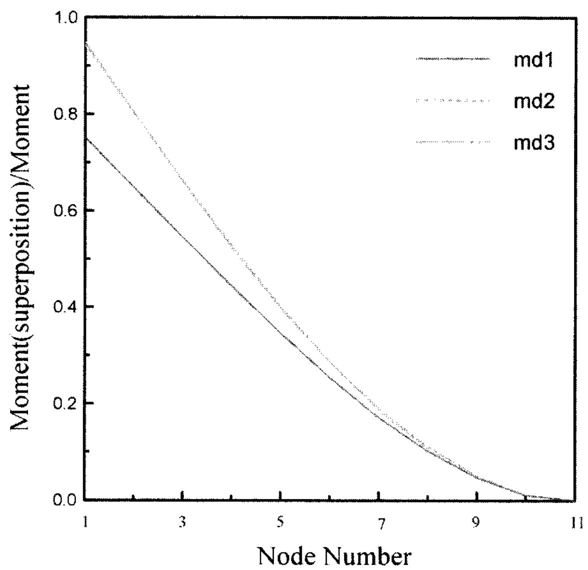

Figure 2. Moment by MD method.

because MA and MT methods contained non-modally represented portion as well as modally represented. For the shear force at each node, MA and MT solutions are shown in Fig. 4 and 5. In these figures, MA and MT methods are more efficient than MD method.

In this example, it is shown that expanded MA and MT methods are well applied to the non-classically damped systems and are more efficient than MD method. In classically damped systems, it would be expected that MT method gives better results overall than MA method

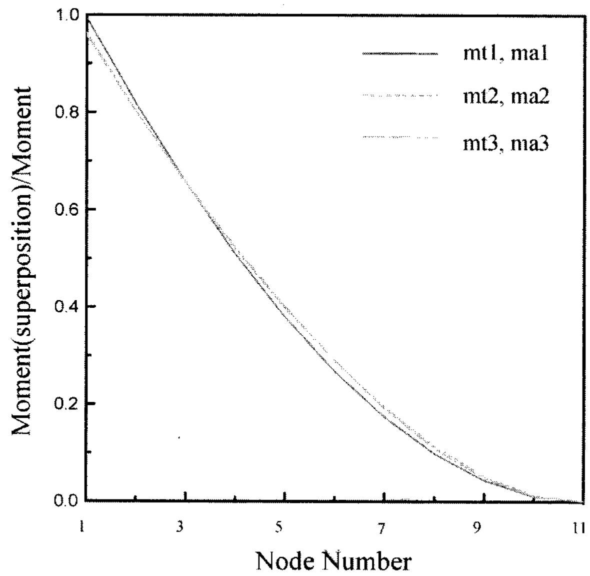

Figure 3. Moment by MD and MT methods.

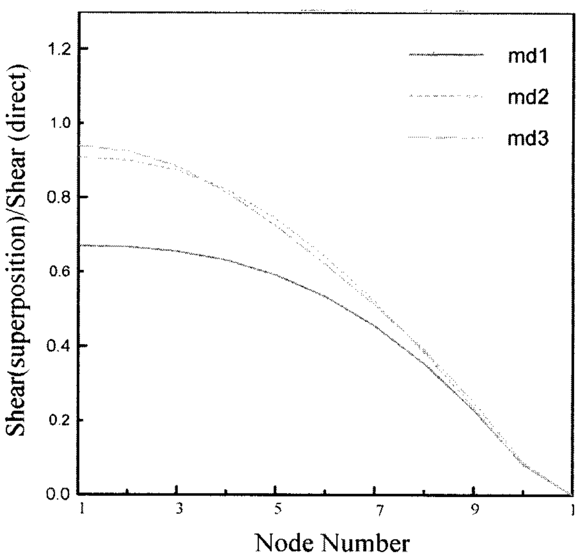

Figure 4. Shear by MD method.

because of the added dynamics. In the non-classically damped systems, however, MT method, when stable, gives the same results as MA method.

6.2. 10-story shear building

This example shows convergent and divergent cases of MT method, respectively. The structure shown in Fig. 6 is a non-classically damped system, which has 10 degrees of freedom with a lumped translational viscous-damper attached at the seventh floor. The complex-conjugate pairs of the eigenvalue are presented in Table 2. Two load cases are considered. For the case 1, input load is given at the tenth floor, and input load is given at the first for the case 2. The time varying portion of each input load, $\sin(\bar{\omega} t)$, is identical and the frequency is 32.0 rad/sec. For each load case, the responses of structures

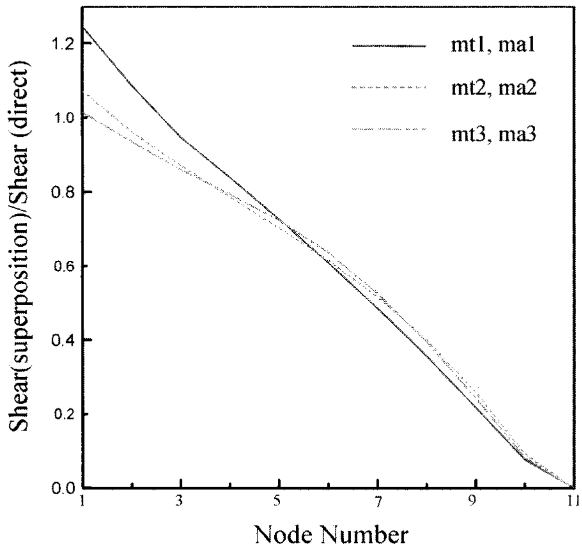

Figure 5. Shear by MA and MT methods.

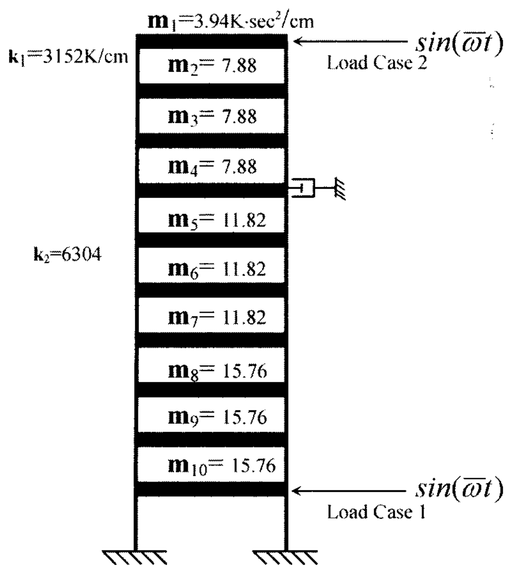

Figure 6. Shear building configuration.

Table 2. Complex eigenvalue set

Mode number	Eigenvalues
1	−0.0316 − 4.0100i
2	−0.0316 + 4.0100i
3	−0.0066 − 10.8381i
4	−0.0066 + 10.8381i
5	−0.0058 − 17.421i
6	−0.0058 + 17.421i

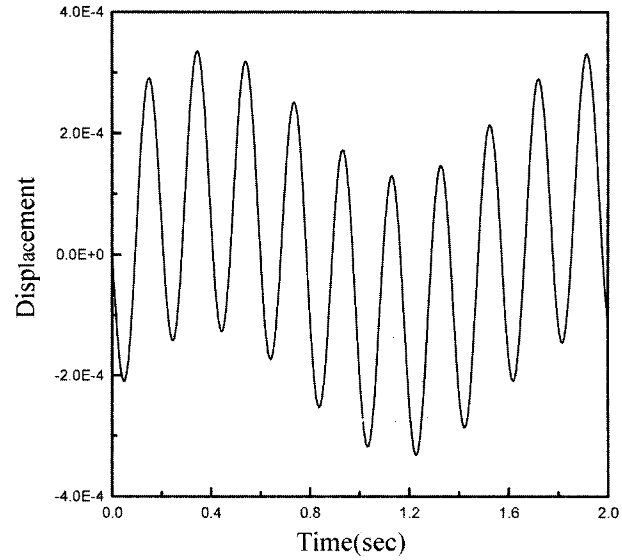

Figure 7. Response by MT method.

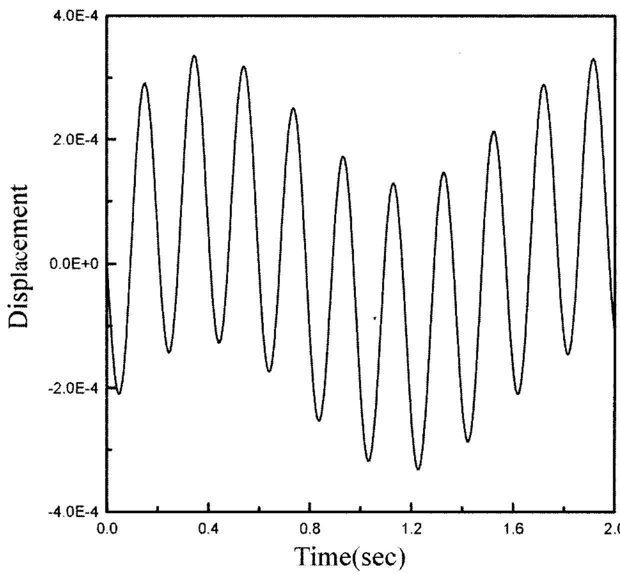

Figure 8. Response by MA method.

are calculated by MA and MT methods, respectively.

For load case 1, the displacements are presented in Figs 7 and 8. Figure 7 shows the displacements by MT method and figure 8 shows the displacement by MA method. In this load case, the value of $s_P = 9.4710 \times 10^3$ satisfies the stability condition, $s_P = \mathbf{P}^T\mathbf{AP} < 0$. Because the stability condition is satisfied, expanded MT method is stable as shown in Fig. 7 and yields same result as expanded MA method in Fig. 8.

For load case 2, the displacements are presented in Figs 9 and 10. Figure 9 shows the displacements by MT method and Fig. 10 shows the displacement by MA method. For load case 2, the value of $s_P = 6.2443 \times 10^3$ does not satisfy the stability condition, $s_P = \mathbf{P}^T\mathbf{AP} < 0$. Therefore, MT method diverges and gives no solution as in Fig. 9, which means that MT method is not applicable to this load case. MA solution, however, is stable and gives the result as in Fig. 10.

From this example, it can be concluded that the applicability of expanded MT method is limited by the stability condition and MT method, when stable, brings

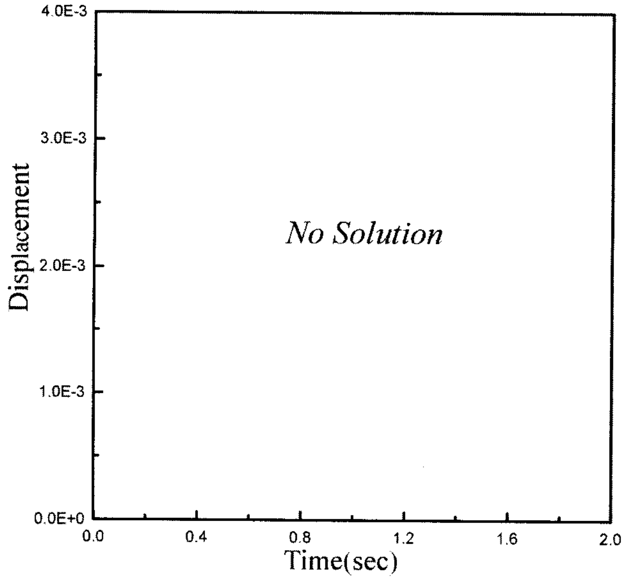

Figure 9. Response by MT method.

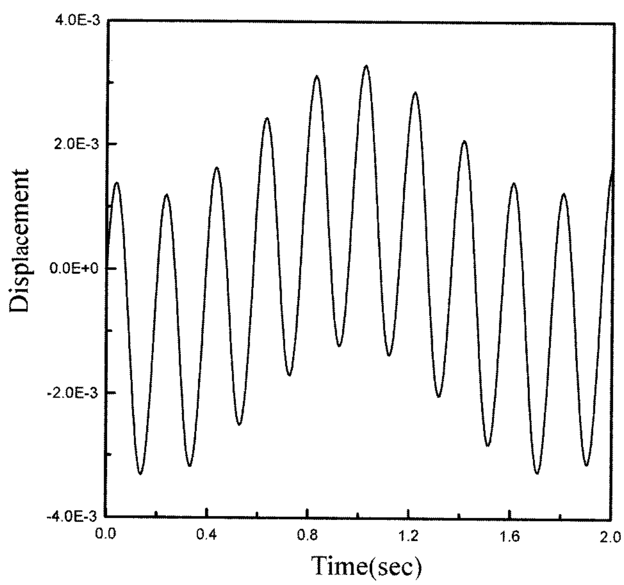

Figure 10. Response by MA method.

out same result as in expanded MA solution.

7. Conclusions

In this paper, MA and MT methods are newly expanded to the non-classically damped systems. The applicability of expansion was verified analytically and numerically.

Expanded MA and MT methods are more efficient than standard MD method. Expanded MT method is conditionally stable depending on the external loading whereas expanded MA method is unconditionally stable. The stability condition, $s_P = \mathbf{P}^T\mathbf{AP} < 0$ of expanded MT method was suggested. In the stable case, MT method has the same results as MA method. Therefore expanded MA method is better than expanded MT method for an analysis of the non-classically damped systems.

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