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Modified Modal Methods for Calculating Eigenpair Sensitivity of Asymmetric Systems

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ABSTRACT: It is well known that many real systems have asymmetric mass, damping and stiffness matrices. In this case, the method for calculating eigenpair sensitivity is different from that of symmetric system. To determine the derivatives of the eigenpairs in asymmetric damped case, a modal method was recently developed by Adhikari. When a dynamic system has many degrees of freedom, only a few lower modes are available, and because the higher modes should be truncated to use the modal method, the errors may become significant. In this paper a procedure for determining the sensitivities of the eigenpairs of asymmetric damped system using a few lowest set of modes is proposed. Numerical examples show that proposed method achieves better calculating efficiency and highly accurate results when a few modes are used.

1 INTRODUCTION

Natural frequency and mode shape of system are essential to understand dynamic behavior of structure. But design parameters can be varied with damage, deterioration, corrosion etc. and this causes variation in natural frequency and mode shape. The variation of eigenpair brings about variation of dynamic behavior of systems and this affects the stability of structure directly. Therefore eigen-sensitivity analysis has played a central part in structural stability analysis and has emerged as an important area of research. And eigenpair sensitivity is used in many areas, the optimization of structure subject to natural frequency, system identification, finite modeling updating, structural control etc.

For symmetric systems, modal methods (Murthy & Haftka 1988, Lim & Junkins 1987) and its modified ones (Wang 1985, Liu et al. 1987) approximate the eigenvector derivatives by a linear combination of the eigenvector. The modal methods employ a modal superposition idea. Therefore, the accuracy is dependent on the number of modes used in calculation. To guarantee the accuracy, the classical modal method needs higher eigenvector. Recently Zeng presented modified modal methods such as multiple modal acceleration and shifted-poles method for the complex eigenvectors in symmetric viscous damping systems, which achieved highly accurate results when only a few modes are used.

However, in many problems in dynamics the inertia, stiffness and damping properties of the system cannot be represented by symmetric matrices. These kind of problems typically arise in the dynamics of actively controlled structures and in many general non-conservative dynamic systems, for example – moving vehicles on roads, missile following trajectories, ship motion in sea water or the study of aircraft flutter. The asymmetric of damping and stiffness terms are often addressed in the context of gyroscopic and follower forces.

To calculate the eigenpair derivatives in this case, Adhikari and M. I. Friwell proposed a modal method by modifying the modal method for symmetric damped systems.

In this paper, by combining the modal method by Adhikari and M. I. Friswell and modal acceleration and shifted-pole method by Zeng modified modal methods for asymmetric damped system are presented. So highly accurate modal method for calculating the derivatives for asymmetric damped systems has been developed. And fewer eigenvectors required for the predetermined accuracy. As a result, the method is more efficient in computation than Adhikari's modal method.

Numerical examples show that proposed method achieves better calculating efficiency and highly accurate results when a few modes are used.

2 PREVIOUS STUDY

2.1 *Modal method for asymmetric systems*

The general equation of motion for an N-degree of freedom system with damping is

$$M\ddot{u}(t)+C\dot{u}(t)+Ku(t)=0 \tag{1}$$

where M, C and K are mass, damping and stiffness matrices, respectively. The traditional restrictions of symmetry and positive definiteness are not imposed on M, C and K, however, it is assumed that M^{-1} exists.

Equation (1) can be rewritten in the following state-space form,

$$A\dot{x}(t)+Bx(t)=0 \tag{2}$$

where

$$A=\begin{bmatrix} C & M \\ M & 0 \end{bmatrix},\ B=\begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix},\ x(t)=\begin{Bmatrix} u(t) \\ \dot{u}(t) \end{Bmatrix}$$

From the equation (2), we obtain following two equations because of the asymmetricity of the system.

$$(sA+B)z=0 \tag{3}$$

$$y^T(sA+B)=0 \tag{4}$$

where s is the eigenvalue, z is the right eigenvector and y is the left eigenvector which is related to the right and left eigenvector of the second order system For distinct eigenvalues, two normalizing conditions by Adhikari are as follows.

$$y_j^T A z_j = 2s_j \tag{5}$$

$$\{u_j\}_{n_j}=\{v_j\}_{n_j} \tag{6}$$

where $\{*_j\}_{n_j}$ denotes the j-th element of a vector n_j is chosen so that the corresponding elements of the eigenvectors are as large as possible. Thus

$$\left|\{u_j\}_{n_j}\right|\left|\{v_j\}_{n_j}\right|=\max_n\left|\{u_j\}_n\right|\left|\{v_j\}_n\right| \tag{7}$$

Differentiate equation (3) with design parameter α. Using this equation, the derivative of eigenvalues can be obtain as

$$s_{j,\alpha}=-\frac{y_j^T(s_jA_{,\alpha}+B_{,\alpha})z_j}{y_j^TAz_j} = -\frac{y_j^T(s_jA_{,\alpha}+B_{,\alpha})z_j}{2s_j} \tag{8}$$

To derive the eigenvector derivatives, we can expand $z_{j,\alpha}$ and $y_{j,\alpha}$ as complex linear combinations of z_i and y_i, for all $i=1,\cdots,2N$.

$$z_{j,\alpha}=\sum_{i=1}^{2N}a_{ji}z_i \tag{9}$$

$$y_{j,\alpha}=\sum_{i=1}^{2N}b_{ji}y_i \tag{10}$$

Differentiate equations (3) and (4) and Substituting the equations (9) and (10) into that equations, we can obtain the coefficient a_{jk} and b_{jk}.

$$a_{jk}=-\frac{y_k^T[s_{j,\alpha}A+s_jA_{,\alpha}+B_{,\alpha}]z_j}{2s_k(s_j-s_k)} \quad \forall k=1,\dots 2N; k\neq j \tag{11}$$

$$b_{jk}=-\frac{y_j^T[s_{j,\alpha}A+s_jA_{,\alpha}+B_{,\alpha}]z_k}{2s_k(s_j-s_k)} \quad \forall k=1,\dots 2N; k\neq j \tag{12}$$

The expressions for a_{jk} and b_{jk} derived above are not valid when k is equal to j. To obtain a_{jj} and b_{jj}, we begin by differentiating equation (5). And substituting the equations (9) and (10) into that equation, gives

$$a_{jj}+b_{jj}=-\frac{y_j^TA_{,\alpha}z_j}{y_j^TAz_j} \tag{13}$$

The second equation is derived by using equation (6). If the n_j-th elements of the left and right eigenvectors remain equal then so do the corresponding elements of the derivatives. Thus

$$b_{jj}-a_{jj}=\frac{1}{\{y_j\}_{n_j}}\sum_{k=1,k\neq j}^{2N}\left[a_{jk}\{z_k\}_{n_j}-b_{jk}\{y_k\}_{n_j}\right] \tag{14}$$

So the derivatives of the eigenvectors are

$$z_{j,\alpha}=\left\{\sum_{k=1,k\neq j}^{N}\left[\frac{z_ky_k^T}{2s_k(s_j-s_k)}+\frac{(z_ky_k^T)^*}{2s_k^*(s_j-s_k^*)}\right]+\frac{(z_jy_j^T)^*}{2s_j^*(s_j-s_j^*)}\right\}f_j+a_{jj}z_j \tag{15}$$

$$y_{j,\alpha}=\left\{\sum_{k=1,k\neq j}^{N}\left[\frac{y_kz_k^T}{2s_k(s_j-s_k)}+\frac{(y_kz_k^T)^*}{2s_k^*(s_j-s_k^*)}\right]+\frac{(y_jz_j^T)^*}{2s_j^*(s_j-s_j^*)}\right\}g_j+b_{jj}y_j \tag{16}$$

where

$$f_j=-(s_{j,\alpha}A+sA_{,\alpha}+B_{,\alpha})z_j, \tag{17}$$

$$g_j=-(s_{j,\alpha}A+sA_{,\alpha}+B_{,\alpha})^Ty_j \tag{18}$$

3 MODIFIED MODAL METHODS

3.1 *Modal acceleration method (MA Method)*

The convergence of equations (15) and (16) is poor. To accurately calculate the eigenvector derivative, the higher modes are required. In a practical situation, there are only some lower modes available, and modal truncation errors will be significant. In response calculations, the modal acceleration approach is used to speed up the convergence and reduce the truncation errors.

At first, we consider the derivatives of right eigenvectors. Separate the response $z_{,\alpha}$ into a pseudostatic response z_{s0} and a dynamic correction response z_{d0}

$$z_{,\alpha} = z_{s0} + z_{d0} \tag{19}$$

where

$$z_{s0} = B^{-1} f \tag{20}$$

$$z_{d0} = z_{,\alpha} - z_{s0} \tag{21}$$

or

$$z_{d0} = (sA + B)^{-1} f - B^{-1} f \tag{22}$$

From the equation (5), we obtain

$$Y^T A Z = \begin{bmatrix} \ddots & & \\ & 2s_j & \\ & & \ddots \end{bmatrix} \tag{23}$$

$$Y^T B Z = \begin{bmatrix} \ddots & & \\ & -2s_j^2 & \\ & & \ddots \end{bmatrix} \tag{24}$$

where Y and Z is the modal matrix to be formed by the right and left eigenvector, respectively.
Substituting the equations (23) and (24) into the equation (22), it yields

$$z_{d0} = Z \begin{bmatrix} \ddots & & \\ & \frac{1}{2s_k(s - s_k)}\left(\frac{s}{s_k}\right) & \\ & & \ddots \end{bmatrix} Y^T f \tag{25}$$

From the equations (20) and (25), we obtain the derivatives of right eigenvectors.

$$\begin{aligned} z_{j,\alpha} = \Bigg\{ B^{-1} + \sum_{k=1,k\neq j}^{N} \Bigg[\left(\frac{s_j}{s_k}\right) \frac{z_k y_k^T}{2s_k(s_j - s_k)} \\ + \left(\frac{s_j}{s_k^*}\right) \frac{(z_k y_k^T)^*}{2s_k^*(s_j - s_k^*)} \Bigg] \\ + \left(\frac{s_j}{s_j^*}\right) \frac{(z_j y_j^T)^*}{2s_j^*(s_j - s_j^*)} \Bigg\} f_j + a_{jj} z_j \end{aligned} \tag{26}$$

By the similar procedure, we can obtain the derivatives of left eigenvectors.

$$\begin{aligned} y_{j,\alpha} = \Bigg\{ B^{-T} \sum_{m=0}^{M-1} (-s_j A^T B^{-T})^m \\ + \sum_{k=1,k\neq j}^{N} \Bigg[\left(\frac{s_j}{s_k}\right)^M \frac{y_k z_k^T}{2s_k(s_j - s_k)} \\ + \left(\frac{s_j}{s_k^*}\right)^M \frac{(y_k z_k^T)^*}{2s_k^*(s_j - s_k^*)} \Bigg] \\ + \left(\frac{s_j}{s_j^*}\right)^M \frac{(y_j z_j^T)^*}{2s_j^*(s_j - s_j^*)} \Bigg\} g_j + b_{jj} y_j \end{aligned} \tag{27}$$

3.2 *Modal acceleration method (MMA Method)*

In the modal acceleration method approach, the convergence of the series is speeded up through preliminary calculation of the pseudostatic response z_{s0} to the excitations f. Based on the similar idea, if the pseudostatic response z_{s1} to the combination force of f and inertia force that comes from the response z_{s0} is preliminarily calculated, the convergence would be further improved for including the effects of the inertia. The pseudostatic response is

$$z_{,\alpha} = z_{s\,n-1} + z_{d\,n-1} \tag{28}$$

where

$$z_{s\,n-1} = B^{-1} f [I - sAz_{s\,n-2}] f \tag{29}$$

$$\begin{aligned} z_{d\,n-1} &= z_{,\alpha} - z_{s\,n-1} \\ &= Z \begin{bmatrix} \ddots & & \\ & \frac{1}{2s_k(s - s_k)}\left(\frac{s}{s_k}\right)^n & \\ & & \ddots \end{bmatrix} Y^T f \end{aligned} \tag{30}$$

By the similar procedure as the modal acceleration approach, the derivatives of right eigenvectors are given as

$$z_{j,\alpha} = \Bigg\{ B^{-1} \sum_{m=0}^{M-1} (-s_j A B^{-1})^m + \sum_{k=1,k\neq j}^{N} \left[\left(\frac{s_j}{s_k}\right)^M \frac{z_k y_k^T}{2s_k(s_j - s_k)} + \left(\frac{s_j}{s_k^*}\right)^M \frac{(z_k y_k^T)^*}{2s_k^*(s_j - s_k^*)} \right] + \left(\frac{s_j}{s_j^*}\right)^M \frac{(z_j y_j^T)^*}{2s_j^*(s_j - s_j^*)} \Bigg\} f_j + a_{jj} z_j \tag{31}$$

By the similarly procedure, the derivatives of left eigenvectors are given as

$$y_{j,\alpha} = \Bigg\{ B^{-T} \sum_{m=0}^{M-1} (-s_j A^T B^{-T})^m + \sum_{k=1,k\neq j}^{N} \left[\left(\frac{s_j}{s_k}\right)^M \frac{y_k z_k^T}{2s_k(s_j - s_k)} + \left(\frac{s_j}{s_k^*}\right)^M \frac{(y_k z_k^T)^*}{2s_k^*(s_j - s_k^*)} \right] + \left(\frac{s_j}{s_j^*}\right)^M \frac{(y_j z_j^T)^*}{2s_j^*(s_j - s_j^*)} \Bigg\} g_j + b_{jj} y_j \tag{32}$$

3.3 *Multiple modal accelerations with Shifted-Poles (MMAS Method)*

For more high convergence rate, the term $(s_j A+B)^{-1}$ is expanded in Taylor's series at the position β as

$$\begin{aligned} (s_j A + B)^{-1} &= [(B + \beta A - (s_j - \beta)(-A)]^{-1} \\ &= (B+\beta A)^{-1}[I + (s_j - \beta)(B+\beta A)^{-1} A]^{-1} \\ &= (B+\beta A)^{-1} \sum_{m=0}^{M-1} [-(s_j - \beta) A (B+\beta A)^{-1}]^m \end{aligned} \tag{33}$$

Through the similar previous procedure using the equation (33), the j-th right eigenvector derivatives are formulated as

$$z_{j,\alpha} = \Bigg\{ (B+\beta A)^{-1} \sum_{m=0}^{M-1} [-(s_j - \beta) A (B+\beta A)^{-1}]^m + \sum_{k=1,k\neq j}^{N} \left[\left(\frac{s_j-\beta}{s_k-\beta}\right)^M \frac{z_k y_k^T}{2s_k(s_j - s_k)} + \left(\frac{s_j-\beta}{s_k^*-\beta}\right)^M \frac{(z_k y_k^T)^*}{2s_k^*(s_j - s_k^*)} \right] + \left(\frac{s_j-\beta}{s_j^*-\beta}\right)^M \frac{(z_j y_j^T)^*}{2s_j^*(s_j - s_j^*)} \Bigg\} f_j + a_{jj} z_j \tag{34}$$

By the similarly procedure, the left eigenvector derivatives is given as

$$y_{j,\alpha} = \Bigg\{ (B+\beta A)^{-T} \sum_{m=0}^{M-1} [-(s_j - \beta) A^T (B+\beta A)^{-T}]^m + \sum_{k=1,k\neq j}^{N} \left[\left(\frac{s_j-\beta}{s_k-\beta}\right)^M \frac{y_k z_k^T}{2s_k(s_j - s_k)} + \left(\frac{s_j-\beta}{s_k^*-\beta}\right)^M \frac{(y_k z_k^T)^*}{2s_k^*(s_j - s_k^*)} \right] + \left(\frac{s_j-\beta}{s_j^*-\beta}\right)^M \frac{(y_j z_j^T)^*}{2s_j^*(s_j - s_j^*)} \Bigg\} g_j + b_{jj} y_j \tag{35}$$

4 NUMERICAL EXAMPLES

Whirling beam whose system matrices are asymmetric is considered as numerical example. This example is a gyroscopic system rotating with high speed and has a lumped mass in center of beam as figure 1.

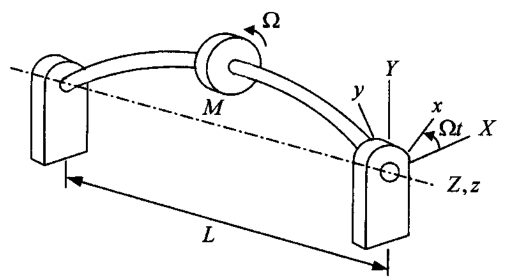

Figure 1. Whirling beam

The equation of motion of gyroscopic system is as follows

$$M\ddot{u}(t) + (C+G)\dot{u}(t) + (K+H)u(t) = F(t)$$

Where M, C, K, F are mass, damping, stiffness and external force matrices respectively, G is gyroscopic matrix and H is circulatory matrix that make system matrix asymmetric. The detail information of this system is represented in Ref. 3.
The numerical values chosen are as follows:

$m_0 = 10kg/m$, $M_0 = 10kg$, $L = 5m$, $EI_x = 4L^3/5\pi^2 Nm^2$

$EI_y = 9L^3/5\pi^2 Nm^2$, $K_1 = K_2 = L^2/20Nm$

$\Omega = \sqrt{21.6\pi}\, rad\, s^{-1}$, $c = h = 1/4Nsm^{-1}$

Number of DOF : 10 Design Parameter : length of beam

Table 1, 2 and 3 shows exact eigenvalues and eigenvectors and their derivatives with respect to length for design variable.

To demonstrate the effectiveness of modified modal methods, right eigenvector derivative is calculated using six modes. Table 4 shows resulting errors. (MMA : M=2, MMAS : β =eigenvalue-1, M=2)

As you can see in the table 4, multiple modal acceleration method with shifted poles is very effective.

To demonstrate the efficiency of MMAS, right eigenvector derivative is calculated using fewer modes. Table 5 shows resulting errors. (β =eigenvalue-1, M=2)

Table 1. Eigenvalues and their derivatives

Mode Number	Eigenvalues	Derivatives
1	-8.4987e-03 +2.3563e+00i	1.3251e-03 +1.5799e+00i
2	-2.7151e-03 +6.3523e+01i	2.2533e-03 +8.5934e-01i
3	1.6771e-02 +1.0548e+01i	3.3394e-03 +3.4034e-01i
4	6.3300e-02 +1.2534e+01i	8.8052e-03 +2.0616e-01i
5	2.3092e-01 +1.4079e+01i	2.1905e-02 +1.7729e-02i
6	-2.6645e-01 +1.5122e+01i	-2.3974e-02 -1.7731e-02i
7	-1.1330e-01 +1.6668e+01i	-8.8052e-03 -2.0608e-01i
8	-5.8579e-02 +1.8650e+01i	-3.7909e-03 -3.3918e-01i
9	-4.7285e-02 +2.2774e+01i	-2.2533e-03 -8.2215e-01i
10	-3.6890e-02 +2.6214e+01i	-1.2833e-03 -1.0644e+00i

Table 2. First right eigenvector and its derivative

DOF Number	Eigenvector	Derivative
1	6.0874e-03 -6.2442e-06i	6.3118e-04 +6.3342e-06i
2	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
3	-7.4415e-03 +6.7358e-06i	-7.6005e-04 -7.1917e-06i
4	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
5	-2.8849e-02 -2.1839e-05i	-6.0508e-03 -1.3386e-05i
6	-1.2500e-05 +1.2110e-02i	1.0624e-05 -4.8858e-03i
7	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
8	+1.4785e-05 -1.4677e-02i	-1.2799e-05 +5.9162e-03i
9	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
10	8.3733e-05 -5.7187e-02i	-3.7941e-05 +1.6957e-02i

Table 3. First left eigenvector and its derivative

DOF Number	Eigenvector	Derivative
1	-6.0874e-03 +6.2442e-06i	-2.2630e-03 +1.8636e-05i
2	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
3	7.4415e-03 -6.7358e-06i	2.6937e-04 -2.3495e-05i
4	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
5	2.8849e-02 +2.1839e-05i	7.0186e-03 -1.0403e-04i
6	-1.2500e-05 -1.2110e-02i	-3.5515e-05 -5.3611e-03i
7	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
8	1.4785e-05 -1.4677e-02i	4.3471e-05 +6.4573e-03i
9	0.0000e+00 +0.0000e+00i	0.0000e+00 +0.0000e+00i
10	8.3733e-05 -5.7187e-02i	1.8429e-04 +1.3508e-02i

Table 4. Error comparison of proposed using four modes

DOF Number	MA	MMA	MMAS
1	8.3063e-01	2.0224e-01	7.1932e-02
2	0.0000e+00	0.0000e+00	0.0000e+00
3	3.8258e+01	1.5063e+00	4.7845e-01
4	0.0000e+00	0.0000e+00	0.0000e+00
5	4.6312e+00	1.2077e-01	3.5371e-02
6	8.0424e-02	5.3198e-02	1.2030e-02
7	0.0000e+00	0.0000e+00	0.0000e+00
8	1.6789e+00	5.8813e-01	1.1871e-01
9	0.0000e+00	0.0000e+00	0.0000e+00
10	5.2018e-01	1.5716e-01	3.0334e-02

Table 5. Error comparison of MMAS using fewer modes

DOF Number	MA	MMA	MMAS
1	7.1932e-02	3.4065e+00	2.0896e+00
2	0.0000+00	0.0000e+00	0.0000e+00
3	4.7845e-01	4.5377e-01	3.1398e+00
4	0.0000e+00	0.0000e+00	0.0000e+00
5	3.5371e-02	3.5167e-02	5.1810e-02
6	1.2030e-02	6.2550e-01	3.8320e-01
7	0.0000e+00	0.0000e+00	0.0000e+00
8	1.1871e-01	1.1407e-01	5.4249e-01
9	0.0000e+00	0.0000e+00	0.0000e+00
10	3.0334e-02	3.0189e-02	3.8830e-02

5 CONCLUSION

The modified modal methods for the eigenpair derivatives of asymmetric damped system has been derived. It is assumed that the system does not possess any repeated eigenvalues. By analyzing the numerical example, it is verified that the proposed methods are more effective than previous method.

For accurate result, previous modal method is needed all eigenvalues and eigenvectors of system. But in practical case, only a few lower modes are available. So the errors may become significant. The modified modal methods is possible to calculate derivatives of eigenvlaues and eigenvectors of asymmetric damped system using a few lower modes.

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