The Second China-Japan-Korea Joint Symposium
on Optimization of Structural and Mechanical Systems (CJK-OSM 2)
Nov. 4 - 8, 2002, Busan, Korea

MODIFIED STURM SEQUENCE PROPERTY FOR DAMPED SYSTEMS

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Abstract

This paper presents a method of checking the number of eigenvalues inside some interested regions on the complex plane or the multiplicity of some complex eigenvalues. In this study, Schur-Cohn matrix is constructed from the coefficients of the characteristic polynomial for the nonproportionally damped system, and factored into its matrix product LDL^T using some standard numerical algorithms. By observing signs of the diagonal elements of the diagonal matrix D, we can determine the number of eigenvalues inside some interested regions on the complex plane, which is very similar to the well-known Sturm sequence property for undamped systems. To verify the applicability of the proposed method, two numerical examples are considered

Introduction

Methods of calculating the number of eigenvalues inside some interested regions or the multiplicity of some eigenvalues have a lot of applications. In control problems, the stability of a dynamic system can be checked and modified by calculating the number of eigenvalues inside some interested regions (Chen 1984; Meirovitch 1990). In eigenvalue analysis problems, by calculating the number of eigenvalues inside some interested regions, we can obtain a shift value to accelerate the convergence of some iterations methods or check missed eigenvalues when the lowest eigenvalues are calculated in the analysis (Hughes 1987; Bathe 1996). For undamped or real

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eigenvalue systems, there has been proposed many theorems and algorithms that can be used for some of the above problems. Among them, the Sturm sequence property is very efficient and stable for numerical implementations (Hughes 1987; Bathe 1996).

In the case of the non-proportionally damped systems such as the soil-structure interaction system, the structural control system and composite structures, however, there are a few studies on a technique to calculate the number of eigenvalues or the multiplicity of some eigenvalues in the literature.

In this paper, Gleyse's theorem (1999), which can count the number of eigenvalues of a characteristic polynomial inside an open unit disk on the complex plane, is extended to calculate the number of eigenvalues inside an open disk of arbitrary radius.

Modified Sturm Sequence Property for Damped Systems

1. Equations of Motion of Damped Systems

In the analysis of dynamic response of structural system, the equation of motion of damped systems can be written as:

$$\mathbf{M}\ddot{u}(t)+\mathbf{C}\dot{u}(t)+\mathbf{K}u(t)=0, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the ($n \times n$) mass, nonclassical damping and stiffness matrices, respectively, and $\ddot{u}(t)$, $\dot{u}(t)$ and $u(t)$ are the ($n \times 1$) acceleration, velocity and displacement vectors, respectively. To find the solution of the free vibration of the system, we consider the following quadratic eigenproblem:

$$\lambda^2\mathbf{M}\phi+\lambda\mathbf{C}\phi+\mathbf{K}\phi=0, \tag{2}$$

in which λ and ϕ are the eigenvalue and eigenvector of the system. There are 2n eigenvalues for the system with n degrees of freedom and these occur either in real pairs or in complex conjugate pairs, depending upon whether they correspond to overdamped or undamped modes.

In general, the mass matrix $\mathbf{M}$ is nonsingular, that is $\det(\mathbf{M}) \neq 0$, and we can reformulate the quadratic system of equation to a state-space form by doubling the order of the system (Meirovitch 1990; Kim and Lee 1999) such as:

$$\mathbf{A}\psi=\lambda\psi,\ \mathbf{A}=\begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix}, \psi=\begin{Bmatrix}\phi \\ \lambda\phi\end{Bmatrix}, \tag{3}$$

The Eq. (3) is a standard eigenproblem, and the form of the matrix $\mathbf{A}$ is very widely used in control engineering field (Meirovitch 1990).

2. Characteristic Polynomial of a Matrix

The characteristic polynomial of Eq. (3) can be represented as:

$$P(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=a_{2n}\lambda^{2n}+a_{2n-1}\lambda^{2n-1}+\cdots+a_1\lambda+a_0=\sum_{i=0}^{2n}a_i\lambda^i, \tag{4}$$

where λ is a complex value and $a_i\ (i=0,1,\ldots,2n)$ are real coefficients. The value of λ that satisfies $P(\lambda)=0$ is called an eigenvalue of the system.

Recently, Rombouts and Heyde (1998) presented an algorithm for calculating the coefficients of the characteristic polynomial of a general square. This algorithm does not include dividing operations, so it is stable and also known as efficient and accurate.

The procedures of Rombouts' algorithm for calculating the coefficients of the characteristic polynomial of a 2n-by-2n matrix **A** are shown in Table I.

Table I. Rombouts' algorithm for calculating characteristic polynomial

***Step 1: Reduce the given matrix* A *(2n-by-2n) to upper Hessenberg Form* $\overline{\mathbf{A}}$.**

- Use Householder reduction or Gauss-elimination like similarity transformations.

Step 2: Initialize a temporary matrix* B *(2n-by-2n).

- Set all the elements of matrix **B** to 0.

Step 3: Calculate elements* b_{ij} *of the matrix* B *using the elements* $\overline{a}_{ij}$ *of the matrix* $\overline{\mathbf{A}}$ *as follows:

DO $j = 2n,1,-1$
    DO $i = 1,j$
        DO $k = 2n-j,\ 1,\ -1$
            $b_{k+1,i} = \overline{a}_{i,j} b_{k,j+1} - \overline{a}_{j+1,j} b_{k,i}$
        ENDDO
        $b_{1,i} = \overline{a}_{i,j}$
    ENDDO
    DO $k = 1,\ 2n-j$
        $b_{k,j} = b_{k,j} + b_{k,j+1}$
    ENDDO
ENDDO

***Step 4: Calculate the coefficients of the characteristic polynomial* $a_i\ (i=0,\cdots,2n)$**

Using the first row of the matrix B, the coefficients of the characteristic polynomial can be computed as: $a_i = (-1)^{2n-i} b_{2n-i,1}$

3. Modified Sturm Sequence Property for Damped Systems

Gleyse (1999) suggested a method of calculating the number of eigenvalues of a real polynomial inside an open unit disk by a determinant representation.

Let $P(\lambda)=\sum_{h=0}^{2n} a_h \lambda_h$ (a_h is a real number) be a characteristic polynomial of a given matrix **A**, then the number of eigenvalues inside an open unit disk can be determined as:

$$N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}] \tag{5}$$

where N_λ is the number of eigenvalues in an open unit disk, $2n$ is the degree of the polynomial, $S[k_0, k_1, k_2, \cdots, k_{2n}]$ is the number of sign changes in the sequence $k_i\ (i=0,1,\cdots,2n)$ and $d_i\ (i=1,2,\cdots,2n)$ is the determinants(minors) of the leading principal submatrices of order i in the Schur-Cohn matrix **T**:

$$\mathbf{T}=\begin{bmatrix} \ddots & & \\ & t_{ij} & \\ & & \ddots \end{bmatrix}, t_{ij}=\sum_{h=0}^{\min(i,j)}\left(a_{2n-i+h}a_{2n-j+h}-a_{i-h}a_{j-h}\right), \tag{6}$$

To apply this theorem to open disks of arbitrary radius ρ, we substitute $\lambda=\rho\bar{\lambda}$ (ρ is a real number) to Equation (4), then the modified characteristic polynomial can be written as:

$$P(\bar{\lambda})=a_{2n}\rho^{2n}\bar{\lambda}^{2n}+\cdots+a_1\rho\bar{\lambda}+a_0=\bar{a}_{2n}\bar{\lambda}^{2n}+\cdots+\bar{a}_1\bar{\lambda}+\bar{a}=\sum_{i=0}^{2n}\bar{a}_i\bar{\lambda}^i, \tag{7}$$

where $\bar{a}_i=a_i\rho^i (i=0,1,\cdots,2n)$ are modified coefficients.

If $\mathbf{T}=\mathbf{LDL}^T$, then:

$$\mathbf{T}_i=\mathbf{L}_i\mathbf{D}_i\mathbf{L}_i^T, \tag{8}$$

where the matrix $\mathbf{T}_i$ is the leading principal submatrices of order i in the Schur-Cohn the matrix $\mathbf{T}$, the matrix $\mathbf{L}_i$ is the leading principal submatrices of order i in the factorized lower triangular matrix $\mathbf{L}$ and the matrix $\mathbf{D}_i$ is the leading principal submatrices of order i in the factorized diagonal matrix $\mathbf{D}$. The value of $d_i (i=1,\cdots,2n)$ can be evaluated as:

$$d_i=\det(\mathbf{T}_i)=\det(\mathbf{L}_i\mathbf{D}_i\mathbf{L}_i^T)=\det(\mathbf{D}_i)=\prod_{h=1}^{i}d_{hh}. \tag{9}$$

4. Numerical Example

To show the effectiveness of the proposed method, a simple spring-mass-damper system that has the exact analytical eigenvalues is considered to verify that the proposed method can exactly calculate the number of eigenvalues in the open disk of arbitrary radius for the eigenproblem with the damping matrix.

Simple Spring-Mass-Damper System

The finite element discretization of the system results in a diagonal mass matrix, a tridiagonal damping and stiffness matrices of the following forms:

$$\mathbf{M}=m\mathbf{I},\ \mathbf{C}=\alpha\mathbf{M}+\beta\mathbf{K}\ \textit{and}\ \mathbf{K}=k\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \tag{10}$$

where α and β are the damping coefficients of the Rayleigh damping. The analytical solutions can be resulted through following relationships:

$$\lambda_{2i-1,2i}=-\xi_i\omega_i\pm j\omega_i\sqrt{1-\xi_i^2},\ \xi_i=\frac{1}{2}\left(\frac{\alpha}{\omega_i}+\beta\omega_i\right)\ \textit{and}\ \omega_i=2\sqrt{\frac{m}{k}}\sin\frac{2i-1}{2n+1}\frac{\pi}{2}\quad \textbf{for}\ i=1,\ldots,n \tag{11}$$

where ω_i and ξ_i are the undamped natural frequency and modal damping ratio, respectively.

A system with order 10 is used in analysis. k and m are 1, and the coefficients, α and β, of the Rayleigh damping are 0.05 and 0.5, respectively. All the eigenvalues and their radius from the origin in the complex plane are as in Table II.

Table II The calculated eigenvalues, coefficients $\bar{a}_i$, diagonal elements d_{ii} of **D**

I	Eigenvalue(λ)		Radius	$\rho=1.005\lvert\lambda_{16}\rvert$ = 1.8109		$\rho=1.005\lvert\lambda_{18}\rvert$ = 1.9207		$\rho=1.005\lvert\lambda_{20}\rvert$ = 1.9875	
	Real	Imag.	($\rho=\lvert\lambda\rvert$)	$\bar{a}_i$	d_{ii}	$\bar{a}_i$	d_{ii}	$\bar{a}_i$	d_{ii}
0	-	-	-	3.079e-04	-	2.055e-04	-	1.618e-00	-
1	-0.0306	-0.1463	0.1495	4.322e-03	1.962e+03	3.059e-03	9.206e+03	2.492e-00	2.239e+04
2	-0.0306	0.1463	0.1495	8.065e-02	1.962e+03	6.055e-02	9.206e+03	5.104e-00	2.239e+04
3	-0.0745	-0.4388	0.4450	6.252e-01	1.962e+03	4.979e-01	9.206e+03	4.343e-00	2.239e+04
4	-0.0745	0.4388	0.4450	4.147e-00	1.962e+03	3.503e-00	9.206e+03	3.161e-00	2.239e+04
5	-0.1585	-0.7133	0.7307	1.947e-00	1.959e+03	1.745e+01	9.204e+03	1.629e+01	2.239e+04
6	-0.1585	0.7133	0.7307	7.601e+01	1.948e+03	7.222e+01	9.187e+03	6.979e+01	2.237e+04
7	-0.2750	-0.9614	1.0000	2.373e+02	1.808e+03	2.391e+02	8.977e+03	2.391e+02	2.212e+04
8	-0.2750	0.9614	1.0000	6.251e+02	1.657e+03	6.681e+02	8.511e+03	6.913e+02	2.135e+04
9	-0.4137	-1.1763	1.2470	1.374e+03	8.707e+02	1.558e+03	6.368e+03	1.668e+03	1.773e+04
10	-0.4137	1.1763	1.2470	2.581e+03	7.447e+02	3.104e+03	5.248e+03	3.439e+03	1.476e+04
11	-0.5624	-1.3540	1.4661	4.111e+03	7.311e+02	5.242e+03	1.799e+03	6.011e+03	6.872e+03
12	-0.5624	1.3540	1.4661	5.610e+03	6.757e+02	7.588e+03	1.459e+03	9.002e+03	5.379e+03
13	-0.7077	-1.4932	1.6525	6.495e+03	-1.056e+02	9.318e+03	1.331e+02	1.144e+04	9.257e+02
14	-0.7077	1.4932	1.6525	6.395e+03	-6.349e-00	9.731e+03	1.098e+02	1.236e+04	7.403e+02
15	-0.8368	-1.5959	1.8019	5.261e+03	2.020e-00	8.491e+03	9.891e+00	1.116e+04	3.517e+01
16	-0.8368	1.5959	1.8019	3.592e+03	1.688e-00	6.149e+03	8.390e+00	8.365e+03	2.792e+01
17	-0.9381	-1.6651	1.9111	1.960e+03	-3.303e-00	3.558e+03	-3.412e+00	5.008e+03	2.735e-00
18	-0.9381	1.6651	1.9111	8.325e+02	-2.537e-00	1.603e+03	-2.312e+00	2.335e+03	2.111e-00
19	-1.0028	-1.7046	1.9777	2.446e+02	3.976e-00	4.996e+02	1.065e+00	7.529e+02	1.622e-00
20	-1.0028	1.7046	1.9777	4.429e+01	2.866e-00	9.595E+01	7.366e+00	1.496e+02	1.208e-00

In this example the checking process performed for open disks of three different radius. First two cases are for checking in between eigenvalues. Their location is selected considering the relative distances between two adjacent eigenvalues except for a conjugate eigenvalues, and two eigenvalues with smaller distances from the next are selected. The third is for checking the number of all the eigenvalues of the system. The radius of the open disk is should be only a bit larger than the selected eigenvalue to ensure that the next eigenvalue is not within the open disk. Jung et al. recommended 1.005 times the magnitude of the largest known eigenvalue. In this example, therefore, the radius ρ of the disk is chosen by 1.005 times the magnitude of the largest eigenvalue($\rho=1.005\lvert\lambda\rvert$). For each cases, the calculated coefficient of the characteristic polynomial $\bar{a}_i$ and diagonal element d_{ii} are as in Table II. Using the signs of d_{ii}, the number of eigenvalues for each cases are calculated as followings:

Case 1: $\rho=1.005\lvert\lambda_{16}\rvert$ = 1.8109, $N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}]$, N_λ = 20 − 4 =16

Case 2: $\rho=1.005\lvert\lambda_{18}\rvert$ = 1.9207, $N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}]$, N_λ = 20 −2 =18

Case 3: $\rho=1.005\lvert\lambda_{20}\rvert$ = 1.9875, $N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}]$, N_λ = 20 − 0 = 20

Referring to Table II, the number of eigenvalues which is inside open disks of radius $\rho=1.005\lvert\lambda_{16}\rvert$

= 1.8109, $\rho = 1.005|\lambda_{18}|$ = 1.9207 and $\rho = 1.005|\lambda_{20}|$ = 1.9875 are 16, 18 and 20 which are exactly agree with the calculated values.

Conclusions

A method of calculating the number of eigenvalues inside an open disk of arbitrary radius or the multiplicity of some interested eigenvalues has been presented. The number of eigenvalues can be checked by observing signs of the diagonal elements of the factorized Schur-Cohn matrix, which is very similar to the well known Strum sequence property.

The proposed method can be theoretically applied to large structures because it is based on mathematically derived theorems and algorithms, however, during calculation of the coefficients of the characteristic polynomial, some small numerical errors are accumulated mainly due to memory limitation. To apply the proposed method to large structures, therefore, further research to reduce the effects of numerical errors should be performed.

Acknowledgements

This research was supported by the National Research Laboratory Grant (No. : 2000-N-NL-01-C-251) in Korea. The financial support is gratefully acknowledged.

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