MATRIX-POWERED LANCZOS ALGORITHM FOR STRUCTURAL EIGENANALYSIS

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Abstract
This paper applies the matrix-powered Lanczos method developed in quantum physics to the eigensolution in structural dynamics. In structural problems, the power technique can be applied to the dynamic matrix. The convergence of the modified Lanczos method using the power of the dynamic matrix is better than that of the conventional Lanczos method. By analyzing numerical examples, the effectiveness of the modified Lanczos method is verified and the optimal power of the dynamic matrix is presented.

Introduction
Lanczos method (Lanczos, 1950) has been known to be very efficient for the eigensolution of structures. To improve the Lanczos method many researchers have studied a variety of procedures. Erricson and Ruhe (1980) have used shifting techniques to accelerate the Lanczos algorithm. Smith et al. (1993) have accelerated the Lanczos method through an implicitly restarted technique. In the fields of quantum physics, Grosso et al. (1993) modified the Lanczos algorithm with power of operator to obtain the eigenstate of quantum systems. Similar power technique is found in accelerated subspace iteration method for structural dynamics (Lam and Bertolini, 1994). While, the modified Lanczos method using the power technique is not applied to structural dynamics yet. This paper applies it to the eigensolution of structural dynamics. In structural eigenproblem, the power technique can be applied to the dynamic matrix $K^{-1}M$. The modified Lanczos method using the power of the dynamic matrix can accelerate the convergence of the conventional Lanczos method. Four numerical examples are presented to verify the effectiveness of the matrix-powered Lanczos method. The optimal power of the dynamic matrix in the method is also presented through numerical examples.

Matrix-Powered Lanczos Method
In the field of quantum physics, Grosso et al. (1993) modified Lanczos recursion by introducing the second power of operator to accelerate the convergence as follows;

$$b_{n+1}f_{n+1} = (H - E_n)^2 f_n - a_n f_n - b_n f_{n-1}$$

(1)
where $H$ is a given operator, $f$ is basis functions, $a$ and $b$ are coefficients and $n$ is Lanczos step number. $E_i$ is trial energy which corresponds to shift in structural dynamics. The concept of power technique in (1) can be applied to the eigenproblem in structural dynamics. The eigenproblem of structure frequently encountered in structural dynamics can be expressed as

$$K\phi_i = \lambda_i M\phi_i \quad (i = 1, 2, \cdots, n) \tag{2}$$

where $M$ and $K$ are symmetric mass and stiffness matrices of order $n$, respectively. $\lambda_i$ and $\phi_i$ are the $i$th eigenpair. To get the solution of (2), Lanczos schemed Ritz bases vectors through Gram-Schmidt orthogonalization of Krylov sequence as follows;

$$x_{i+1} = (K\mu^{-1}M)^i x_0 - \sum_{j=1}^{i} \nu_j x_j \tag{3}$$

where $x_0$ is a starting vector, $x_j$ is $j$th Lanczos vector, $\nu_j$ is the component of $v_i$ along $x_j$, $K\mu = K - \mu M$ and $\mu$ is shift. The concept of power technique can be applied to the dynamic matrix in (3), then following modified Gram-Schmidt orthogonalization can be introduced

$$x_{i+1} = ((K\mu^{-1}M)^{\delta}) x_0 - \sum_{j=1}^{i} \nu_j x_j \tag{4}$$

where $\delta$ is positive integer. (4) means that an approximated eigenvector, whose number of iteration is $\delta i$, is contained in $(i+1)$ Lanczos vectors. Whereas, in (3), $(i+1)$ Lanczos vectors contain an approximated eigenvector whose number of iterations is $i$. Therefore, (4) gives a better solution than (3). From (4), modified Lanczos recursion can be derived as

$$\tilde{x}_i = (K\mu^{-1}M)^{\delta} x_i - \alpha_i x_i - \beta_{i-1} x_{i-1} \tag{5}$$

where $\alpha_i$ and $\beta_i$ are scalar coefficients obtained by

$$\alpha_i = x_i^T M (K\mu^{-1}M)^{\delta} x_i, \quad \beta_i = (\tilde{x}_i^T M \tilde{x}_i)^{1/2} \tag{6}$$

then the next Lanczos vector is

$$x_{i+1} = \tilde{x}_i / \beta_i \tag{7}$$

$X = [x_1 \ x_2 \ \cdots \ x_q]$ leads to the triagonalized standard eigenproblem of reduced order $q$ ($<< n$)

$$T\tilde{\phi}_i = \left(1/(\lambda_i - \mu)^{\delta}\right)\tilde{\phi}_i \quad (i = 1, 2, \cdots, q) \tag{8}$$

where

$$T = X^T M (K\mu^{-1}M)^{\delta} X = \begin{bmatrix} \alpha_1 & \beta_1 & & & & \\ \beta_1 & \alpha_2 & \beta_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \alpha_{q-1} & \beta_{q-1} & & \\ & & & \beta_{q-1} & \alpha_q \end{bmatrix} \tag{9}$$

The number of total operations for the matrix-powered Lanczos algorithm is
\[ N_{\text{total}} = (1/2)nm^2 + (q^2 + 4q \delta + 5q + 3/2)nm \]

\[ + ((3/2)q^2 + q \delta + (7/2)q)n + 10q^2 + q + \sum_{j=2}^{q} 6js_j \]  \hspace{1cm} (10)

where \( n \) is system order, \( m \) half-bandwidth, \( q \) the number of calculated Lanczos vectors and \( s_j \) the number of iterations of \( j \)th step in QR iteration for the eigenvalues of tridiagonal system.

**Numerical Examples**

A simple spring-mass system (Chen, 1993), a plane framed structure (Bathe and Wilson, 1972), a three-dimensional frame structure (Bathe and Wilson, 1972) and a three-dimensional building frame (Kim and Lee, 1999) are analyzed to verify the effectiveness of the matrix-powered Lanczos method. With the predetermined error norm of \( 10^{-6} \), the number of operations for calculating desired eigenpairs is compared. To examine the optimal power of the dynamic matrix, numerical examples are analyzed with varying power of the dynamic matrix. The geometric configurations and the material properties of the example structures are shown in Figs. 1 ~ 4.

![Simple spring-mass system (DOFs: 100)](image1)

*Fig. 1. Simple spring-mass system (DOFs: 100)*

![Plane framed structure (DOFs: 330)](image2)

*Fig. 2. Plane framed structure (DOFs: 330)*
Some results are shown in Table 1 and Fig. 5. The 1st power (δ = 1) corresponds to the conventional Lanczos method. Table 1 and Fig. 5 show that the convergence of the matrix-powered Lanczos method is better than that of the conventional Lanczos method. However, in some cases, high matrix power causes failure in convergence due to the numerical instability.
Table 1. Number of operations for calculating desired eigenpairs

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<th>Structure</th>
<th>No. of eigenpairs</th>
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*: Failure in convergence due to numerical instability

Fig. 5. Number of operations for calculating desired eigenpairs
Conclusions
This paper applies a power technique to the Lanczos method for the eigenproblem solution of structures. The characteristics of the matrix-powered Lanczos method by the numerical results from examples are summarized as follows:
(1) Since the power of the dynamic matrix in the matrix-powered Lanczos method can reduce the number of required Lanczos vectors, the convergence of the matrix-powered Lanczos method is better than that of the conventional Lanczos method.
(2) The optimal power of the dynamic matrix that reduces the number of operations and gives numerically stable solution in the matrix-powered Lanczos method is the second power.

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References


