ALGEBRAIC METHOD FOR EIGENPAIR DERIVATIVES OF DAMPED SYSTEM WITH REPEATED EIGENVALUES

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ABSTRACT : A simplified method for the computation of derivatives of eigenvalues and eigenvectors associated with repeated eigenvalues is presented. Adjacent eigenvectors and orthonormal conditions are used to compose an algebraic equation. The algebraic equation developed can be used to compute derivatives of both eigenvalues and eigenvectors simultaneously. Since the coefficient matrix in the proposed algebraic equation is non-singular, symmetric and based on *N*-space, it is numerically stable and very efficient compared to previous methods. To verify the efficiency of the proposed method, the finite element model of the cantilever beam is considered.

KEYWORDS : Sensitivity, damped system, eigenpair, repeated eigenvalues, derivatives of eigenvalues, derivatives of eigenvectors

1. INTRODUCTION

Methods for computing eigenvalue and eigenvector derivatives have been studied by many researchers in the past 30 years. The importance of obtaining sensitivities for eigenvalue problems stems from the fact that partial derivatives with respect to system parameters are extremely important for effecting efficient design modifications for given situations, for gaining insight into the reasons for discrepancies between structural analyses and dynamic tests by varying its design parameters, and for indicating system model changes that will improve correlations between analyses and tests.

The sensitivity of eigenvalue problem with repeated eigenvalues has been a focus of recent interest. The most common circumstances under which repeated eigenvalues or nearly equal eigenvalues occur in typical structural or mechanical systems are instances where system symmetry exists, such as structures with two or more planes of reflective or cyclic symmetric or in the limiting case of axisymmetric bodies or certain reasons. In this case, since the eigenspace spanned by eigenvectors corresponding to repeated eigenvalues is degenerate, any linear combination of eigenvectors can be a eigenvector. For the eigenvector derivative to be found, the adjacent eigenvectors which lie “adjacent” to the m(multiplicity of repeated eigenvalue) distinct eigenvectors appearing when a design parameter varies must be calculated first. To do so, the approximate eigenvectors could be varied continuously by varying the design parameter.

For the real symmetric case, a generalization of Nelson’s method was obtained by Ojalvo and amended by Mills-Curren and Dailey. These methods are lengthy and complicated for finding eigenvector derivatives and clumsy for programming, because they basically follow Nelson’s algorithm. Lee *et al.*(1999) developed an analytical method that give exact solutions while it maintains N-space, but it finds eigenvalue derivative from classical method as before.

In this paper, an efficient algebraic method for the eigenpair sensitivities of damped systems with repeated eigenvalues is presented. Contrary to previous methods, the proposed method finds the eigenvalue and eigenvector sensitivities simultaneously from one equation. The proposed method doesn’t use a state space

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equation(2N-space), instead of it, the method maintains N-space because a singularity problem is solved by using only one side condition. The algebraic equation of the proposed method may be efficiently solved by the LDL^T decomposition method. If the derivatives of the stiffness, mass and damping matrices can be analytically found, the proposed method can find the exact eigenpair derivatives. And it only requires the corresponding eigenpair information differently from modal methods.

2. EIGENPAIR SENSITIVITY IN DAMPED SYSTEMS

When an eigenvalue has multiplicity *m* and a design parameter is perturbed, the corresponding eigenvectors may split into as many as *m* distinct eigenvectors. Since the eigenspace spanned by eigenvectors corresponding to repeated eigenvalues is degenerate, any linear combination of eigenvectors can be an eigenvector. For the eigenvector derivative to be found, the adjacent eigenvectors which lie "adjacent" to the *m*(multiplicity of repeated eigenvalues) distinct eigenvectors appearing when a design parameter varies, must be calculated first. Otherwise, the eigenvectors would jump discontinuously with a varying design parameter. The derivatives of these adjacent eigenvectors are represented in this section.

The eigenvalue problem of a damped system can be expressed as

$$(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})\phi = \mathbf{0} \tag{1}$$

where M, C and K are the matrices of mass, damping and stiffness, respectively, and these are ($n\times n$) symmetric matrices. M is positive definite and K is positive definite or semi-positive definite. The first step in finding derivatives of eigenvalues and eigenvectors of repeated eigenvalues is to find corresponding adjacent eigenvectors. Suppose that all eigenpairs are known and multiplicity of the eigenvalue λ_m is *m*. Define eigenvalue problem as:

$$\mathbf{M}\Phi_m\Lambda_m^2 + \mathbf{C}\Phi_m\Lambda_m + \mathbf{K}\Phi_m = \mathbf{0} \tag{2}$$

where

$$\Lambda_m = \lambda_m\mathbf{I}_m \text{ and } \Phi_m = [\phi_{\mathbf{i+1}}\ \phi_{\mathbf{i+2}}\ \cdots\ \phi_{\mathbf{i+}m}] \tag{3}$$

where Φ_m ($n\times m$) is the matrix of eigenvectors corresponding to the repeated eigenvalues. $\mathbf{I_m}$ is the identity matrix of order *m* and λ_m is the eigenvalue of multiplicity m for the eigenspace spanned by the columns of Φ_m. The orthonormal condition for the (i+1)th eigenvector is as follows:

$$\phi_{\mathbf{i+1}}^{\mathbf{T}}(2\lambda_{\mathbf{i+1}}\mathbf{M} + \mathbf{C})\phi_{\mathbf{i+1}} = \mathbf{1} \tag{4}$$

Since the multiplicity is *m*, the orthonormal condition for the matrix Φ_m is as follows:

$$\Phi_m^{\mathbf{T}}(2\lambda_m\mathbf{M} + \mathbf{C})\Phi_m = \mathbf{I}_m \tag{5}$$

Adjacent eigenvectors can be expressed in terms of Φ_m by an orthogonal transformation such as

$$\mathbf{X}_m = \Phi_m\mathbf{T} \tag{6}$$

where T is an orthonormal transformation matrix and its order *m*;

$$\mathbf{T}^{\mathbf{T}}\mathbf{T} = \mathbf{I}_m \tag{7}$$

The columns of the matrix $\mathbf{X}_m$ are the adjacent eigenvectors for which a derivative can be defined. It is natural that the adjacent eigenvectors also satisfy the orthonormal condition:

$$\mathbf{X}_m^{\mathbf{T}}(2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X}_m = \mathbf{T}^{\mathbf{T}}\Phi_m^{\mathbf{T}}(2\lambda_m\mathbf{M} + \mathbf{C})\Phi_m\mathbf{T} = \mathbf{T}^{\mathbf{T}}\mathbf{T} = \mathbf{I}_m \tag{8}$$

The next procedure is to find T and then to find $\mathbf{X}_m$. If design parameter α varies, the derivatives of the matrix Λ_m with respect to α can be expressed as

$$\Lambda_{m,\alpha} = \mathbf{diag}\left(\lambda_{\mathbf{i+1},\alpha}, \lambda_{\mathbf{i+2},\alpha}, \cdots, \lambda_{\mathbf{i+}m,\alpha}\right) \quad \textbf{(9)}$$

where $(\circ)_{,\alpha}$ represents the derivative of $(\circ)$ with respect to the design parameter α.
Consider following another eigenvalue problem to find $\mathbf{X}_m$.

$$\mathbf{MX}_m\Lambda_m^2 + \mathbf{CX}_m\Lambda_m + \mathbf{KX}_m = \mathbf{0} \quad \textbf{(10)}$$

where the order of adjacent eigenvector matrix $\mathbf{X}_m$ is (n×m) and the order of eigenvalue matrix $\mathbf{\Lambda_m}$ is (m×m). Differentiating the above eigenvalue problem with respect to the design parameter α, and rearranging yields:

$$(\lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K})\mathbf{X}_{m,\alpha} + (2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X}_m\Lambda_{m,\alpha} = -(\lambda_m^2\mathbf{M}_{,\alpha} + \lambda_m\mathbf{C}_{,\alpha} + \mathbf{K}_{,\alpha})\mathbf{X}_m \quad \textbf{(11)}$$

Pre-multiplying at each side of equation (11) by $\mathbf{\Phi_m^T}$ and substituting $\mathbf{X_m} = \mathbf{\Phi_m T}$ into it gives a new eigenvalue problem such as

$$\mathbf{DT} = \mathbf{ET}\Lambda_{m,\alpha} \quad \textbf{(12)}$$

where

$$\mathbf{D} = \Phi_m^{\mathbf{T}}(\lambda_m^2\mathbf{M}_{,\alpha} + \lambda_m\mathbf{C}_{,\alpha} + \mathbf{K}_{,\alpha})\Phi_m, \quad \textbf{(13)}$$

$$\mathbf{E} = -\Phi_m^{\mathbf{T}}(2\lambda_m\mathbf{M} + \mathbf{C})\Phi_m = -\mathbf{I}_m \quad \textbf{(14)}$$

The orthogonal transformation matrix T can be obtained by solving **equation (12)**, and then the adjacent eigenvectors by relation $\mathbf{X}_m = \Phi_m\mathbf{T}$.
The proposed method starts with the equations of the derivative of the eigenvalue problem composed of the system matrices and the adjacent eigenvectors, **equation (11)**, and the orthonormal condition, **equation (8)**. Differentiating **equation (8)** with respect to the design parameter α gives

$$\mathbf{X}_m^{\mathbf{T}}(2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X}_{m,\alpha} + \mathbf{X}_m^{\mathbf{T}}\mathbf{MX}_m\Lambda_{m,\alpha} = -\mathbf{0.5}\mathbf{X}_m^{\mathbf{T}}(2\lambda_m\mathbf{M}_{,\alpha} + \mathbf{C}_{,\alpha})\mathbf{X}_m \quad \textbf{(15)}$$

Since the unknown or interested values are $\mathbf{X}_{m,\alpha}$ and $\Lambda_{m,\alpha}$, **equation (11)** and **equation (15)** can be combined into a single matrix form as follows:

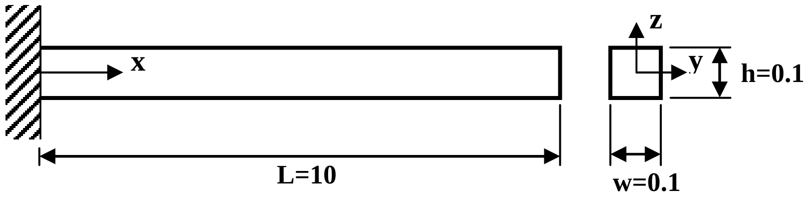
$$\begin{bmatrix} \lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K} & (2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X}_m \\ \mathbf{X}_m^{\mathbf{T}}(2\lambda_m\mathbf{M} + \mathbf{C}) & \mathbf{X}_m^{\mathbf{T}}\mathbf{MX}_m \end{bmatrix} \begin{Bmatrix} \mathbf{X}_{m,\alpha} \\ \Lambda_{m,\alpha} \end{Bmatrix} = \begin{bmatrix} -(\lambda_m^2\mathbf{M}_{,\alpha} + \lambda_m\mathbf{C}_{,\alpha} + \mathbf{K}_{,\alpha})\mathbf{X}_m \\ -\mathbf{0.5}\mathbf{X}_m^{\mathbf{T}}(2\lambda_m\mathbf{M}_{,\alpha} + \mathbf{C}_{,\alpha})\mathbf{X}_m \end{bmatrix} \quad \textbf{(16)}$$

where the order of coefficient matrix on the left side of **equation (16)** is (n+m)×(n+m) and the matrix of the right side of equation is (n+m)×m. The first-order derivatives $\mathbf{X}_{m,\alpha}$ and $\Lambda_{m,\alpha}$ can be found by solving **equation (16)**.

Contrary to previous method, the sensitivities of the eigenvalue and eigenvector can be obtained simultaneously from one augmented equation. It maintains N-space without use of state space equation and finds the eigenpair derivatives simultaneously. The proposed method requires only corresponding eigenpair information differently from modal methods, and gives exact solution and guarantees numerical stability.

3. NUMERICAL EXAMPLE

As an illustrative example in case of the proportionally damped system with repeated eigenvalues, a cantilever beam with square cross section is considered as in Figure 1. This is FEM model composed of 20 elements and 21 nodes. Each node has four degrees of freedom. The structure has 80 degrees of freedom. The design parameter is taken as the beam width w.



Number of nodes: 21
Number of elements: 20
Number of DOF: 80

Young's Modulus: E=2.10×10^{11}
Mass density: ρ=7.85×10^3 kg/m^3
Design parameter: width of beam

Figure 1. Cantilever beam

The calculated lowest 12 eigenvalues and their derivatives of the cantilever beam are listed in the second and third columns of Table 1. The derivatives of the repeated eigenvalues are different in that one is close to zero while the other is not. Since the design parameter is the width of the beam, the repeated eigenvalues are split into distinct ones as the cross-section of the beam is no longer square after changing the width of the beam. The actual and approximate eigenvalues of the changed system with $\Delta w/w$=0.01 are represented in the second and third columns of **Table 2**. The next two columns in Table 2 are the variations of eigenvalues and eigenvectors between initial and changed ones and the last two are the errors of the approximations. The errors are reasonably smaller than the corresponding variations, and one can say that the proposed method gives good results for the case of repeated eigenvalues and for a proportionally damped system.

Table 1. The lowest 12 eigenvalues of the initial cantilever beam and results of the sensitivity analysis

Mode number	Eigenvalues	Derivatives of eigenvalues
1, 2	-1.4279e-03 ±j5.2496e+00	-2.8057e-10 ∓j3.5347e-10
3, 4	-1.4279e-03 ±j5.2496e+00	-2.2756e-02 ±j5.2494e+01
5, 6	-5.4154e-02 ±j3.2895e+01	-6.6265e-10 ±j2.3445e-10
7, 8	-5.4154e-02 ±j3.2895e+01	-1.0818e+00 ±j3.2886e+02
9, 10	-4.2409e-01 ±j9.2090e+01	6.9247e-10 ∓j6.9600e-10
11, 12	-4.2409e-01 ±j9.2090e+01	-8.4753e+00 ±j9.2029e+02

Table 2. The lowest 12 eigenvalues and approximated eigenvalues of the changed cantilever beam, variations of eigenpairs and errors of approximations

Mode number	Initial System		Changed System		Error of Approximation	
	Eigenvalue	Eigenvalue Derivative	Eigenvalue	Approximated Eigenvalue	Eigenvalue	Eigenvector
1,2	-1.4279-03 ∓ j5.2496e-00	-2.8057e-10 ±j3.5347e-10	-1.4279e-03 ∓ j5.2496e-00	-1.4279e-03 ∓ j5.2496e-00	2.2283e-11	3.7376e-05
3,4	-1.4279e-03 ∓ j5.2496e-00	-2.2756e-02 ∓ j5.2494e+01	-1.4556e-03 ∓ j5.3021e-00	-1.4555e-03 ∓ j5.3021e-00	2.6622e-08	1.0000e-04
5,6	-5.4154e-02 ∓ j3.2895e+01	-6.6265e-10 ∓ j2.3445e-10	-5.4154e-02 ∓ j3.2895e+01	-5.4154e-02 ∓ j3.2895e+01	3.6899e-12	3.7376e-05
7,8	-5.4154e-02 ∓ j3.2895e+01	-1.0818e+00 ∓ j3.2886e+02	-5.5241e-02 ∓ j3.3224e+01	-5.5236e-02 ∓ j3.3224e+01	1.6763e-07	1.0001e-04
9,10	-4.2409e-01 ∓ j9.2090e+01	6.9247e-10 ±j6.9600e-10	-4.2409e-01 ∓ j9.2090e+01	-4.2409e-01 ∓ j9.2090e+01	9.1432e-12	3.7376e-05
11,12	-4.2409e-01 ∓ j9.2090e+01	-8.4753e+00 ∓ j9.2029e+02	-4.3261e-01 ∓ j9.3010e+01	-4.3256e-01 ∓ j9.3010e+01	4.6508e-07	1.0002e-04

4. CONCLUSIONS

A simple algorithm for the calculation of eigenpair derivatives of the damped system with repeated eigenvalues is presented. The proposed method finds derivatives of eigenvalues and eigenvectors simultaneously from one augmented equation by solving a stable linear algebraic equation. This approach avoids the use of the state space equation and considers the damping problem explicitly by introducing a side condition of derivatives of normalization condition. Therefore, computation for the equation with N-order can be maintained and the computer storage and analysis time required of the proposed method are smaller than those of previous methods.

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