Proceedings of the International Conference on Structural Dynamics, Vibration, Noise and Control, Hong Kong, December 5-7, 1995

ISBN 962-367-188-1

An Efficient Free Vibration Anaysis of Structures with Multiple or Close Natural Frequencies

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Abstract An efficient numerical method which can analyze the eigenproblem for the large structural system with multiple or close eigenvalues is presented. This method is formulated by applying the accelerated Newton-Raphson method to eigenproblem obtained from solution of a constrained stationary value problem. This method can calculate the natural frequencies and mode shapes without any numerical instability which may be often encountered in the well-known methods such as the subspace iteration method or the determinant search method. The efficiency of this method is verified by comparing convergence and solution time for numerical examples with those of the well-known methods.

1. Introduction

The analysis of structures for dynamic loads is of considerable importance in many fields of engineering. If the dynamic analysis is performed by the mode superposition method, the eigenproblem must be first solved.

If structures with multiple or close natural frequencies, such as multi-span bridge containment building of nuclear power plant, cable-stayed bridge, tire and the structure whose cross-section is symmetric are analyzed by the subspace iteration method[1] or the determinant search method[1] which has been mainly used for solving eigenproblems, the numerical instability or the slow convergence may be often encountered.

The objective of this paper is to present an efficient solution method in order to improve numerical stability and increase convergence in case of structures with multiple or close natural frequencies. Numerical examples are presented to show the efficiency of the proposed method.

2. Method of Analysis

The following generalized eigenvalue problem is considered in dynamic analysis.

$$A\bar{x}_i = \lambda_i B\bar{x}_i \qquad (i = 1, 2, 3, \ldots, n) \tag{1}$$

where A and B are the stiffness matrix and the mass matrix of order n, respectively. B is assumed to be positive definite and A positive semidefinite. λ_i is the ith natural frequency squared and $\bar{x}_i$ the corresponding mode shape.

The eigenvalue problem with s multiple or close eigenvalues can be written as follows.

$$A\bar{x}_j = \lambda_j B\bar{x}_j \quad (j=m,m+1,\ldots,m+s-1) \tag{2}$$

Let us take s vectors $\bar{y}_j (j=m,m+1,\ldots,m+s-1)$ which are orthonormal with respect to B and in the neighborhood of the subspace spanned by the eigenvectors $\bar{x}_j (j=m,m+1,\ldots,m+s-1)$.

Then, Eq.(2) can be written as Eq.(3) using the constrained stationary value problem.[2]

$$A\bar{y}_j = BY\bar{d}_j \quad (j=m,m+1,\ldots,m+s-1) \tag{3}$$

where $Y = [\bar{y}_m, \bar{y}_{m+1}, \cdots, \bar{y}_{m+s-1}]$

$\bar{d}_j^T = (\mu_{mj}, \mu_{m+1,j}, \cdots, \mu_{m+s-1,j}) \quad (j=m,m+1,\ldots,m+s-1)$

and μ_{ij} is the approximate value of a eigenvalue λ_j.

Let us assume that initial approximate solutions of Eq. (3), $\bar{d}_j^{(0)}$ and $\bar{y}_j^{(0)}$ ($j=m, m+1, \ldots, m+s-1$), are available. Denote an approximate eigenvalue and the corresponding eigenvector after k iterations by $\bar{d}_j^{(k)}$ and $\bar{y}_j^{(k)}$ ($k = 0, 1, 2, \cdots$). Then, we have

$$\bar{r}_j^{(k)} = A\,\bar{y}_j^{(k)} - B\,Y^{(k)}\,\bar{d}_j^{(k)} \quad (j=m,m+1,\ldots,m+s-1) \tag{4}$$

where the residual vector $\bar{r}_j^{(k)}$ is not generally zero because of substitution of approximate values into Eq. (3).

In order to make the residual vector a null vector, the Newton-Raphson technique is applied.

$$\bar{r}_j^{(k+1)} = 0 = A\,\bar{y}_j^{(k+1)} - B\,Y^{(k+1)}\,\bar{d}_j^{(k+1)} \quad (j=m,m+1,\ldots,m+s-1) \tag{5}$$

where $\bar{d}_j^{(k+1)} = \bar{d}_j^{(k)} + \Delta\bar{d}_j^{(k)} \quad (j=m,m+1,\ldots,m+s-1)$ (6)

$$\bar{y}_j^{(k+1)} = \bar{y}_j^{(k)} + \Delta\bar{y}_j^{(k)} \quad (j=m,m+1,\ldots,m+s-1) \tag{7}$$

in which $\Delta\bar{d}_j^{(k)}$ and $\Delta\bar{y}_j^{(k)}$ are incremental values for $\bar{d}_j^{(k)}$ and $\bar{y}_j^{(k)}$.

Introducing Eqs.(6) and (7) into Eq. (5) and neglecting the small nonlinear terms, we obtain the linear equations for $\Delta\bar{d}_j^{(k)}$ and $\Delta\bar{y}_j^{(k)}$ [2],

$$-A\bar{y}_j^{(k)} + BY^{(k)}\bar{d}_j^{(k)} = A\Delta\bar{y}_j^{(k)} - \mu_{jj}^{(k)}B\Delta\bar{y}_j^{(k)} - BY^{(k)}\Delta\bar{d}_j^{(k)} \quad (j=m,m+1,\ldots,m+s-1) \tag{8}$$

Because there are only n equations with $n+s$ unknowns, s components of $\bar{d}_j^{(k)}$ and n components of $\bar{y}_j^{(k)}$, a side condition must be introduced for the solution of Eq. (8). The side condition to arrive at a set of $n+s$ equations with $n+s$ unknowns is

$$(Y^{(k)})^T\,B\,\Delta\bar{y}_j^{(k)} = 0 \quad (j=m,m+1,\ldots,m+s-1) \tag{9}$$

This means that the allowable changes in the approximate eigenvector are orthogonal to the latest subspace spanned by the approximate eigenvectors with respect to the mass matrix. This prevents unlimited drift in the eigenvector which is, after all, not determined in magnitude.

Writing Eqs. (8) and Eq. (9) in matrix form, we get

$$\begin{bmatrix} A-\mu_{jj}^{(k)}B & -BY^{(k)} \\ -Y^{(k)T}B & 0 \end{bmatrix}\begin{Bmatrix} \Delta\bar{y}_j^{(k)} \\ \Delta\bar{d}_j^{(k)} \end{Bmatrix} = -\begin{Bmatrix} \bar{r}_j^{(k)} \\ 0 \end{Bmatrix} \qquad (j=m,m+1,...,m+s-1) \tag{10}$$

The coefficient matrix for the incremental values, $\Delta\bar{d}_j^{(k)}$ and $\Delta\bar{y}_j^{(k)}$, is of order $n+s$, symmetric and nonsingular.[2]

The above algorithm using the Newton-Raphson method has a rapid convergence. On the other hand, it is expensive and inconvenient since a new coefficient matrix has to be formed and refactorized in each iteration step. The shortcoming of the above algorithm may be avoided by using the following modified Newton-Raphson method in Eq. (10).

$$\begin{bmatrix} A-\mu_{jj}^{(0)}B & -BY^{(k)} \\ -Y^{(k)T}B & 0 \end{bmatrix}\begin{Bmatrix} \Delta\bar{y}_j^{(k)} \\ \Delta\bar{d}_j^{(k)} \end{Bmatrix} = -\begin{Bmatrix} \bar{r}_j^{(k)} \\ 0 \end{Bmatrix} \qquad (j=m,m+1,...,m+s-1) \tag{11}$$

The coefficient matrix in Eq. (11) is also symmetric and nonsingular.[2]

The convergence rate in $\bar{d}_j^{(k+1)}$ and $\bar{y}_j^{(k+1)}$ of Eq. (11) using the modified Newton-Raphson method can be written as

$$\kappa_j^{(k+1)} = h^2\,\kappa_j^{(k)} \tag{12}$$

$$\theta_j^{(k+1)} = h\,\theta_j^{(k)} \tag{13}$$

where $\kappa_j^{(k+1)} = \left|\dfrac{\mu_{jj}-\mu_{jj}^{(k)}}{\mu_{jj}}\right|$, $h = \max_{i\neq j}\left|\dfrac{\lambda^*-\mu_{jj}^{(0)}}{\lambda_i-\mu_{jj}^{(0)}}\right|$ and $\lambda^* = \lambda_m = \lambda_{m+1} = ... = \lambda_{m+s-1}$.[2]

Once the submatrix $A-\mu_{jj}^{(0)}B$ in the coefficient matrix of Eq. (11) is decomposed into the LDL^T, where L is lower triangular and D is diagonal, only a small number of operations are required for the solution of Eq. (11) in the succeeding iterations, since the vector $B\,\bar{y}_j^{(k)}$ in the coefficient matrix is changed in each iteration. However, due to negligence of the small nonlinear term $(\mu_{jj}^{(k+1)}-\mu_{jj}^{(0)})\,B\,\Delta\bar{y}_j^{(k)}$, the convergence is lower, which in turn increases the number of iterations for a solution. The above scheme has been presented by Lee and Robinson.[2]

Some of these drawback may be avoided by the following procedures.

$$\bar{y}_j^{(k+1)} = \bar{y}_j^{(k)} + \alpha_j^{(k)}\Delta\bar{y}_j^{(k)} \qquad (j=m,m+1,...,m+s-1) \tag{14}$$

$\alpha_j^{(k)}$ in Eq. (14) is a value to minimize the norm of the residual vector, $\bar{r}_j^{(k+1)}$, and can be evaluated by using the least square method as follows.

$$\frac{\partial}{\partial\alpha_j^{(k)}}\{(\bar{r}_j^{(k+1)})^T\,\bar{r}_j^{(k+1)}\} = 0 \qquad (j=m,m+1,...,m+s-1) \tag{15}$$

$$\alpha_j^{(k)} = -\frac{(\Delta\bar{y}_j^{(k)})^T(A-\mu_{jj}^{(k+1)}B)(A-\mu_{jj}^{(k+1)}B)\bar{y}_j^{(k)} - (\Delta\bar{y}_j^{(k)})^T(A-\mu_{jj}^{(k+1)}B)\left(\sum_{\substack{i=m \\ i\neq j}}^{m+s-1}\mu_{ij}^{(k+1)}B\bar{y}_i^{(k)}\right)}{(\Delta\bar{y}_j^{(k)})^T(A-\mu_{jj}^{(k+1)}B)(A-\mu_{jj}^{(k+1)}B)\Delta\bar{y}_j^{(k)}} \tag{16}$$

Note that $\mu_{ij}^{(k+1)}$ and $\Delta\bar{y}_j^{(k)}$ have been obtained by Eq. (11).

If the order of the system is n, and the bandwidths of the stiffness matrix and mass matrix are m_a and m_b, respectively, the number of operations for evaluating $\alpha_j^{(k)}$ in the first iteration step is $2nm_a+2nm_b+(s+5)n+1$. This is large compared to $(s+4)nm_a+2snm_b+(1/2)n(s^2+7s+4)$ which is required in each iteration step in Eq. (11). However, only the number of $7n+1$ operations is required to evaluate $\alpha_j^{(k)}$ after the 2nd iteration, which is negligible, because we use computational results in the previous iteration. Thus, solution time of the proposed method is decreased by improving convergence.

As k increases, the incremental values $\Delta\bar{d}_j^{(k)}$ and $\Delta\bar{y}_j^{(k)}$ will vanish. Then from Eq. (11)

$$\lim_{k\to\infty}\bar{r}_j^{(k)}=\lim_{k\to\infty}(A\bar{y}_j^{(k)}-BY^{(k)}\bar{d}_j^{(k)})=0 \quad (j=m,m+1,\ldots,m+s-1) \tag{17}$$

Eq. (17) can be written as

$$AY=BYD \tag{18}$$

where $Y=[\bar{y}_m,\bar{y}_{m+1},\ldots,\bar{y}_{m+s-1}]$, $D=[\bar{d}_m,\bar{d}_{m+1},\ldots,\bar{d}_{m+s-1}]$

If the eigenvalues λ_j $(j=m,m+1,\ldots,m+s-1)$are multiple, the values of the off-diagonal elements of D are all zero and its diagonal elements have an equal value which is the desired multiple eigenvalue. Moreover, the vectors in Y are the corresponding eigenvectors. If the eigenvalues are merely close, additional operations are required because the values of the off-diagonal elements of D are not zero.[2]

3. Numerical Examples

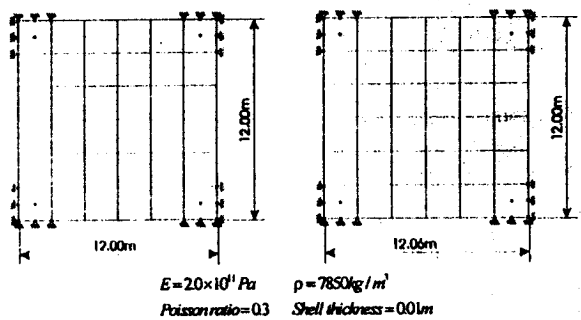
The simply-supported plane and the cooling tower are analyzed to verify the efficiency of the proposed method. By using three methods separately, the subspace iteration method, the determinant search method and the proposed method, each convergence and solution time(CPU time) used to calculate 10 eigenpairs with error norm of 1.E-09 are compared. Especially to get the best results we applied the accelerated scheme to the subspace iteration method. Intermediate results with relative error of 1.E-01 in the subspace iteration method are used as initial values of the proposed method. $\alpha_j^{(k)}$ is applied to the eigenpair whose error norm is over 1.E-01. All runs are executed in the IRIS4D-20-S17.

3.1 Simply-Supported Plane Structure

The simply-supported plane structures shown Fig. [1] consist of 36 nine-node shell elements, 169 nodes and 701 degrees-of-freedom. The stiffness matrix and mass matrix have the mean half-bandwidths of 89.

3.1.1 multiple natural frequencies

The simply-supported plane structure with multiple natural frequencies is shown in Fig. [1-a]. $\alpha_j^{(k)}$ is applied to the 6th, the 8th and the 10th eigenpair with error norm exceeding 1.E-01.



(a) multiple natural frequencies (b) close natural frequencies

Fig. [1] the simply-supported plane[3]

Each solution time for three methods to have 10 eigenpairs with the error norm of 1.E-09 is summarized in Table [1]. If we let the solution time for the proposed method be 1, it takes 1.59 times for the accelerated subspace iteration method, 6.49 times for the determinant search method. For each solution method the convergence of the 8th eigenpair to which $\alpha_j^{(k)}$ is applied is represented in Fig. [3].

3.1.2 close natural frequencies

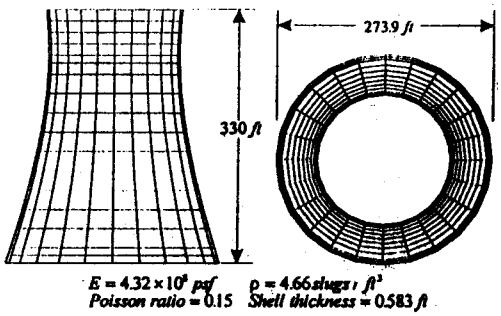
The simply-supported plane structure with close natural frequencies is shown in Fig. [1-b]. $\alpha^{(k)}$ is applied to the 7th, the 8th and the 10th eigenpair with error norm exceeding 1.E-01.

Each solution time for three methods to have 10 eigenpairs with the error norm of 1.E-09 is summarized in Table [1]. If we let the solution time for the proposed method be 1, it takes 1.65 times for the accelerated subspace iteration method, 4.69 times for the determinant search method. For each solution method the convergence of eigenpairs to which $\alpha_j^{(k)}$ is applied is represented from Fig. [4] to Fig. [5].

3.2 Cooling Tower Structure

The cooling tower structure shown Fig. [2] consists of 408 four-node shell elements, 432 nodes and 2448 degrees-of-freedom. The stiffness and the mass matrix have the mean half-bandwidths of 201. $\alpha_j^{(k)}$ is applied to the 8th and the 10th eigenpair with error norm exceeding 1.E-01.

Each solution time for two solution methods to have 10 eigenpairs with the error norm of 1.E-09 is summarized in Table [1]. Determinant search method is not applied because it did not give us the good results. If we let the solution time for the proposed method be 1, it takes 2.02 times for the accelerated subspace iteration method. For each solution method the convergence of eigenpairs to which $\alpha_j^{(k)}$ is applied is represented from Fig. [6] to Fig. [7]



(a) Elevation (b) Plan

Fig. [2] Cooling Tower[4]

4. Conclusions

This paper proposes an efficient numerical method for the multiple or close eigenvalue problems using the accelerated Newton-Raphson method. As shown in the examples, the proposed method has the characteristics as follows.

① The proposed method is a general technique which can identify the eigenpairs of a structure efficiently without any numerical instability in case of multiple or close eigenvalues as well as distinct eigenvalues.

② The proposed method will not be affected by the eigenpairs previously calculated, because each eigenpair is essentially obtained independently.

③ The decrement of convergence has been found in some cases using the accelerated Newton-Raphson method because it uses a constant step length without considering the shape of characteristic function. With this proposed method, the disadvantage stated above can be solved by taking the adequate step length($\alpha_j^{(k)}$) which includes the shape of the characteristic function to minimize the norm of the residual vector in each iteration step.

References

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Table [1] Solution Times(CPU Times, sec)

	Simply-Supported Plane(ratio)		Cooling Tower(ratio)
	Multiple	Close	
Proposed Method	172.39 (1.00)	177.41 (1.00)	3067.71 (1.00)
Acc. Subspace Iteration Method	274.60 (1.59)	291.85 (1.65)	6182.49 (2.02)
Determinant Search Method	1118.60 (6.49)	832.09 (4.69)	-

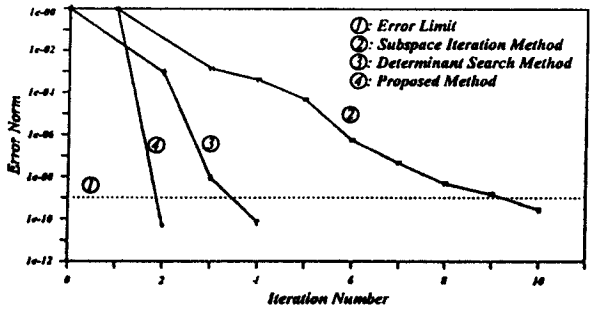

Fig. [3] Convergence of the 8th Eigenpair

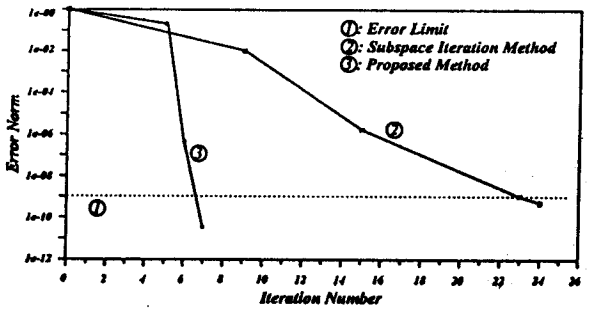

Fig. [6] Convergence of the 8th Eigenpair

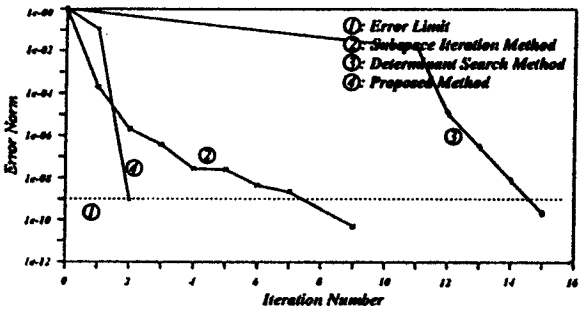

Fig. [4] Convergence of the 7th Eigenpair

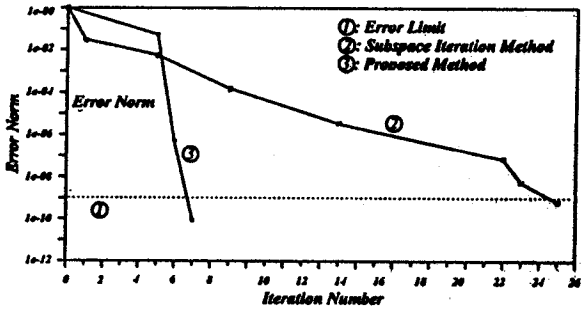

Fig. [7] Convergence of the 10th Eigenpair

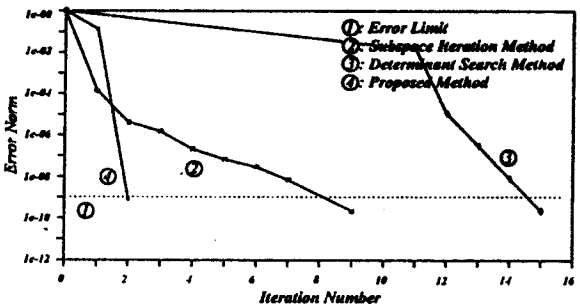

Fig. [5] Convergence of the 8th Eigenpair