A Very Efficient Solution Technique for the Natural Frequencies and Mode Shapes of Large Structures by Accelerated Newton-Raphson Method

*In-Won Lee

Professor, Department of Civil Engineering, KAIST, Taejon, Korea

Man-Cheol Kim, Hyung-Jo Jung

Graduate Student, Department of Civil Engineering, KAIST, Taejon, Korea

Arthur R. Robinson

Professor Emeritus, Department of Civil Engineering, Univ. of Illinois at Urbana-Champaign, Urbana, Illinois, USA

Abstract

A very efficient numerical method which can calculate the natural frequencies and mode shapes for large structural systems is presented. The method applies the accelerated Newton-Raphson method to eigenproblems. If eigenvalues are not multiple, the method can calculate the natural frequencies and mode shapes without any numerical instability which may be often encountered in the inverse iteration method with shift. The efficiency of the method is verified by comparing convergence and solution time for numerical examples with those of the subspace iteration method and the determinant search method.

1. INTRODUCTION

The analysis of a number of physical problems requires the solution of an eigenproblem. It is therefore natural that with the increased use of computational methods operating on discrete representations of physical problems, the development of efficient techniques for the calculation of eigenvalues and eigenvectors has attracted much attention. In particular, the use of finite element techniques can lead to large systems of equations, and the efficiency of an overall response analysis can depend to a significant degree on the effectiveness of the solution of the required eigenvalues and eigenvectors.

The determinant search method[1 4,6] and the subspace iteration method[1 4,7 9] have been mainly used for solving such eigenproblems. The determinant search method is a method which combines the polynomial iteration method, the Sturm-sequence method and the vector iteration method. The method can be efficiently used in the analysis of systems with small bandwidth, since the matrix decomposition must be executed at each step[6]. The subspace iteration method is a method which combines the simultaneous inverse iteration method and the Rayleigh-Ritz method. The method has been used mostly, but the following shortcomings have been identified after extensive use of the method[8,9].

1. When the inverse iteration with shift is applied to the method to increase convergence, the shift value may be close to an exact eigenvalue, and then numerical instability may be encountered during triangularization process.

2. The solution time used in the subspace iteration method rises rapidly as the number of eigenpairs considered is increased. It is due to a number of factors that can be neglected when the solution of only a few eigenpairs is required.

   (a) When p (p : the number of eigenpairs to be required) is large, the convergence rate of the eigenvector, ϕ_p, can be close to one.

   (b) If q (q : the number of iteration vectors) increases, so does the number of operations per subspace iteration significantly.

   (c) If q increases, the convergence of the smallest eigenvalues is generally achieved in a few iterations, and the converged vectors plus $(p+1)$th to qth iteration vectors are only included in the additional iterations to provide solution stability and to accelerate the convergence of the large required eigenvalues.

Lee and Robinson[11] proposed an solution method to improve numerical stability and increase convergence. To further improve the method, the accelerated Newton-Raphson method is proposed here. As examples for calculating eigenvalues and the corresponding mode shapes, one plane frame and one three-dimensional building frame structure are analyzed to prove the efficiency of the proposed method.

2. METHOD OF ANALYSIS

Problem Definition . - Consider a generalized eigenproblem such as,

$$K\phi_j = \lambda_j M\phi_j \quad (j = 1, 2, 3, \cdots, n) \tag{1}$$

$$\phi_j^T M\phi_j = 1 \tag{2}$$

where K and M are the stiffness matrix and mass matrix of order n, respectively. M is assumed to be positive definite and K positive semidefinite. λ_j is the jth natural frequency squared and ϕ_j the corresponding mode shape. The objective is to solve for the p lowest eigenvalues and the associated eigenvectors.

Newton-Raphson Method .[10] - Let us assume that initial approximate solutions of eqn(1), $\lambda_j^{(0)}$ and $\phi_j^{(0)}$ are available. Denote an approximate eigenvalue and the corresponding eigenvector after k iterations by $\lambda_j^{(k)}$ and $\phi_j^{(k)}$ $(k = 0, 1, 2, \cdots)$. Then, we have

$$r_j^{(k)} = K\phi_j^{(k)} - \lambda_j^{(k)} M\phi_j^{(k)} \tag{3}$$

$$(\phi_j^{(k)})^T M\phi_j^{(k)} = 1 \tag{4}$$

where the residual vector, $r_j^{(k)}$, is not generally zero because of substitution of approximate values into eqn(1). In order to get a approximate solution converged to the eigenvalue and the corresponding eigenvector of the system, the residual vector should be removed. Let us apply the Newton-Raphson method for this purpose.

$$\begin{aligned} r_j^{(k+1)} &= K\phi_j^{(k+1)} - \lambda_j^{(k+1)} M\phi_j^{(k+1)} \\ &= 0 \end{aligned} \tag{5}$$

$$(\phi_j^{(k+1)})^T M\phi_j^{(k+1)} = 1 \tag{6}$$

where

$$\lambda_j^{(k+1)} = \lambda_j^{(k)} + \Delta\lambda_j^{(k)} \tag{7}$$

$$\phi_j^{(k+1)} = \phi_j^{(k)} + \Delta\phi_j^{(k)} \tag{8}$$

Substituting eqns(3), (4), (7) and (8) into eqns(5), (6) and neglecting the higher order terms, $(\Delta\lambda_j^{(k)} M \Delta\lambda_j^{(k)})$ and $((\Delta\phi_j^{(k)})^T M \Delta\phi_j^{(k)})$, we get

$$(K - \lambda_j^{(k)} M)\Delta\phi_j^{(k)} - \Delta\lambda_j^{(k)} M \phi_j^{(k)} = -r_j^{(k)} \tag{9}$$

$$(\phi_j^{(k)})^T M \Delta\phi_j^{(k)} = 0 \tag{10}$$

where $\Delta\lambda_j^{(k)}$ and $\Delta\phi_j^{(k)}$ are unknown incremental values of $\lambda_j^{(k)}$ and $\phi_j^{(k)}$.

Writing eqns(9) and (10) in matrix form, we get

$$\begin{bmatrix} K - \lambda_j^{(k)} M & -M\phi_j^{(k)} \\ -(\phi_j^{(k)})^T M & 0 \end{bmatrix} \begin{Bmatrix} \Delta\phi_j^{(k)} \\ \Delta\lambda_j^{(k)} \end{Bmatrix} = -\begin{Bmatrix} r_j^{(k)} \\ 0 \end{Bmatrix} \tag{11}$$

The coefficient matrix for the incremental values is of order $n+1$ and symmetric. If λ_j's are not multiple, it is nonsingular.[11] If the shift is near an eigenvalue, numerical stability problems in the inverse iteration method with shift can be encountered. However, Newton-Raphson method resolves the above problems, which is the main difference compared with the inverse iteration method with shift.

The above algorithm using the Newton-Raphson method, despite of its rapid convergence, is not efficient, since a new coefficient matrix has to be formed and refactorized in each iteration step.

Modified Newton-Raphson Method .[11] - The complete elimination procedure in each iteration may be avoided by using the modified Newton-Raphson method in eqn(11) as follows :

$$\begin{bmatrix} K - \lambda_j^{(0)} M & -M\phi_j^{(k)} \\ -(\phi_j^{(k)})^T M & 0 \end{bmatrix} \begin{Bmatrix} \Delta\phi_j^{(k)} \\ \Delta\lambda_j^{(k)} \end{Bmatrix} = -\begin{Bmatrix} r_j^{(k)} \\ 0 \end{Bmatrix} \tag{12}$$

The coefficient matrix in eqn(12) is both nonsingular and symmetric.

Convergence rates of $\lambda_j^{(k)}$ and $\phi_j^{(k)}$ in eqn(12) using the modified Newton-Raphson method can be written as[11]

$$\gamma_j^{(k)} = h^2 \gamma_j^{(k-1)} \tag{13}$$

$$\theta_j^{(k)} = h\theta_j^{(k-1)} \tag{14}$$

where $\gamma_j^{(k)} = |\frac{\gamma_j - \gamma_j^{(k)}}{\gamma_j}|$ and $h = max|\frac{\lambda_j - \lambda_j^{(0)}}{\lambda_i - \lambda_j^{(0)}}|$. $\theta_j^{(k)}$ represents the angle between $\phi_j^{(k)}$ and ϕ_j, that is, an error in $\phi_j^{(k)}$. As shown in eqns(13) and (14), the convergence rate of eigenvalue, $\gamma_j^{(k)}$, is quadratic in h and that of eigenvector, $\theta_j^{(k)}$, linear in h.

Once the submatrix $(K - \lambda_j^{(0)}M)$ is decomposed into the LDL^T (L : lower triangular matrix, D : diagonal matrix), a small number of operations are required for the solution of eqn(12) in the succeeding iterations, since the vector $M\phi_j^{(k)}$ in the coefficient matrix is only changed in each iteration. However, due to negligence of the small nonlinear term $(\lambda_j^{(k+1)} - \lambda_j^{(0)})M\Delta\phi_j^{(k)}$, the convergence is lower. Thus, the number of iterations for a solution increases. The above scheme has been presented by Lee and Robinson.[11]

Accelerated Newton-Raphson Method . - To further improve the eigenvector, the accelerated scheme is proposed here, that is,

$$\begin{bmatrix} K - \lambda_j^{(0)}M & -M\phi_j^{(k)} \\ -(\phi_j^{(k)})^T M & 0 \end{bmatrix} \begin{Bmatrix} \Delta\phi_j^{(k)} \\ \Delta\lambda_j^{(k)} \end{Bmatrix} = - \begin{Bmatrix} r_j^{(k)} \\ 0 \end{Bmatrix} \tag{15}$$

$$\lambda_j^{(k+1)} = \lambda_j^{(k)} + \Delta\lambda_j^{(k)} \tag{16}$$

$$\phi_j^{(k+1)} = \phi_j^{(k)} + \alpha_j^{(k)}\Delta\phi_j^{(k)} \tag{17}$$

Fig. 1 The Accelerated Newton-Raphson Method

$\alpha_j^{(k)}$ is a step length to minimize the norm of the residual vector. It can be evaluated by using the least square technique as follows;

$$\frac{\partial}{\partial\alpha_j^{(k)}}(r_j^{(k+1)})^T r_j^{(k+1)} = 0 \tag{18}$$

$$\alpha_j^{(k)} = -\frac{(\Delta\phi_j^{(k)})^T(K-\lambda_j^{(k+1)}M)(K-\lambda_j^{(k+1)}M)\phi_j^{(k)}}{(\Delta\phi_j^{(k)})^T(K-\lambda_j^{(k+1)}M)(K-\lambda_j^{(k+1)}M)\Delta\phi_j^{(k)}} \tag{19}$$

Note that $\lambda_j^{(k+1)}$ and $\Delta\phi_j^{(k)}$ have been obtained by eqns(16) and (15), respectively.

Operation Count and Summary of Algorithm . - In order to obtain an estimate of the cost to solve an eigenproblem, consider the number of Central Processor operations required for solution. The actual cost includes, of course, the cost of the Peripheral Processor time. However, this time is system and programming dependent and is therefore not considered in this investigation.

Let one operation equal one multiplication which nearly always is followed by an addition. Assume that the half bandwidths of K amd M, i.e., m_K and m_M, are full. A summary of the steps in the accelerated Newton-Raphson method together with the corresponding number of operations is given in the Table 1.

Table 1 Operation Count for Accelerated Newton-Raphson Method

Calculation	Number of Operations
$K-\lambda_{j_j^{(0)}}^{(0)}M$	$n(m_M+1)$
$LDL^T = K-\lambda_j^{(0)}M$	$\frac{1}{2}nm_K(m_K+3)$
Iteration $k=1,2,\cdots$	
$k=1$	
$K\phi_j^{(k)}$	$n(2m_K+1)$
$M\phi_j^{(k)}$	$n(2m_M+1)$
$k=2,\cdots$	
$K\phi_j^{(k)} = K(\phi_j^{(k-1)}+\alpha_j^{(k-1)}\Delta\phi_j^{(k-1)})$	n
$M\phi_j^{(k)} = M(\phi_j^{(k-1)}+\alpha_j^{(k-1)}\Delta\phi_j^{(k-1)})$	n
$r_j^{(k)} = (K-\lambda_j^{(k)}M)\phi_j^{(k)}$	n
$LDL^T = M\phi_j^{(k)}$	$n(m_K+1)$
Solve eqn(15) for $\Delta\lambda_j^{(k)}$ and $\Delta\phi_j^{(k)}$	$2n(m_K+1)$
Solve eqn(19) for $\alpha_j^{(k)}$	$2nm_K+2nm_M+6n$
$\phi_j^{(k+1)} = \phi_j^{(k)}+\alpha_j^{(k)}\Delta\phi_j^{(k)}$	n
$\lambda_j^{(k+1)} = \lambda_j^{(k)}+\Delta\lambda_j^{(k)}$	0
Number of Operations for $k=1$	$7nm_K+4nm_M+13n$
Number of Operations for $k=2,\cdots$	$5nm_K+2nm_M+13n$

The number of operations for evaluating $\alpha_j^{(k)}$ in the first iteration step is $2nm_K+2nm_M+7n$. This is large compared to $5nm_K+2nm_M+6n$ which is required in each iteration step in the modified Newton-Raphson method. However, the number of $7n$ operations is more required in all the way after the 2nd iteration because we can use computational results in the previous step, which is negligible. Thus, solution time of the proposed method decreases.

Numerical Stability.[11] - The most remarkable characteristic of the accelerated Newton-Raphson mehtod is numerical stability. The numerical stability can be proved by identifying the nonsingularity of the coefficient matrix of eqn(15).

Let the coefficient matrix of eqn(15) be denoted by $C^{(k)}$, that is

$$C^{(k)} = \begin{bmatrix} K - \lambda_j^{(0)} M & -M\phi_j^{(k)} \\ -(\phi_j^{(k)})^T M & 0 \end{bmatrix} \tag{20}$$

The determinant of $C^{(k)}$ is a continuous function of the approximate eigenvalue and eigenvectors. Hence, If $C^{(k)}$ is nonsingular when the approximate value in $C^{(k)}$ becomes the exact one, then it will be also nonsingular for close enough approximations. The resulting matrix C will be

$$C = \begin{bmatrix} K - \lambda_j M & -M\phi_j \\ -(\phi_j)^T M & 0 \end{bmatrix} \tag{21}$$

To prove the nonsingularity of C, we introduce the following eigenproblem

$$CU = M^* U D \tag{22}$$

where

$$M^* = \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix} \tag{23}$$

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_{n+1} \end{bmatrix} \tag{24}$$

$$D = diag\begin{pmatrix} \tau_1 & \tau_2 & \cdots & \tau_{n+1} \end{pmatrix} \tag{25}$$

The eigenpairs of the eigenproblem eqn(20) are as follows

- Eigenvector u : $\left\{ \begin{matrix} \phi_i \\ 1 \end{matrix} \right\}, \left\{ \begin{matrix} \phi_i \\ -1 \end{matrix} \right\}, \left\{ \begin{matrix} \phi_k \\ 0 \end{matrix} \right\}, \ k = 1, 2, \ldots, n, k \neq j$ (26)
- Eigenvalue τ : $-1 \ , \ 1 \ , \ (\lambda_k - \lambda_j) \ , \ k = 1, 2, \ldots, n, k \neq j$ (27)

Considering the determinant of eqn(22)

$$\begin{aligned} det[C] &= det[M^*]det[D] \\ &= -det[M]\prod_{k=1,k\neq j}^{n}(\lambda_k - \lambda_j) \end{aligned} \tag{28}$$

The determinant of C is not zero because of $det[M] \neq 0$ by definition. The nonsingularity of the matrix $C^{(k)}$ is shown. That is, the numerical stability of the proposed method is proved.

Missed Eigenvalues . - Some of the eigenvalues and corresponding eigenvectors of interest may be missed when the initial approximations are not suitable. In order to check whether this occurs, the Sturm-sequence property[3] may be applied. A computed eigenvalue can be checked using the above property with negligible extra computation, since the decomposition of the matrix $(K - \lambda_j^{(0)}M)$ has already been carried out during the procedure for the solution of eqn(12). If some of the eigenvalues of interest are detected to be missing, the solutions can be found by the accelerated Newton-Raphson method.

3. NUMERICAL EXAMPLES

The plane frame and the three-dimensional building frame used by K. J. Bathe[7] are analyzed to verify the efficiency of the proposed method. With the predetermined error norm of 10^{-9}, the structures are analyzed by three different methods; the subspace iteration method, the determinant search method and the proposed method, where the error norm[3] is computed by

$$error\ norm = \frac{\|(K - \lambda_j^{(k)}M)\phi_j^{(k)}\|_2}{\|K\phi_j^{(k)}\|_2} \tag{29}$$

Each convergence rate and solution time(CPU time) used to calculate 15 eigenpairs are compared. Intermediate results with relative error of 10^{-1} in the subspace iteration method are taken as initial values for the proposed method. The relative error[3] in the subspace iteration method is computed as follows

$$relative\ error = \left|\frac{\lambda_j^{(k+1)} - \lambda_j^{(k)}}{\lambda_j^{(k+1)}}\right| \tag{30}$$

$\alpha_j^{(k)}$ is applied to the eigenpair whose error norm is over 10^{-1}. All runs are executed in the IRIS4D-20-S17 with 10 Mips and 0.9 Mflops.

Plane Frame Structure

Fig. 2 Plane Frame Structure

The plane frame structure which has 10 stories and 10 bays shown in Fig. 2 consists of 210 beam elements, 121 nodes and 330 degrees-of-freedom. The mean half bandwidths of both the stiffness matrix and mass matrix are 30. $\alpha_j^{(k)}$ is applied to the 13th, the 14th and the 15th eigenpair with error norm exceeding 10^{-1}.

Solution times for three methods are summarized in Table 2. If we let the solution time for the proposed method be 1, it takes 2.7 times for the subspace iteration method, 2.3 times for the determinant search method. For each solution method, the convergence of eigenpairs to which $\alpha_j^{(k)}$ is applied is depicted in Figs. 3 to 4. From the figures it is obvious that the convergence of the proposed method is superior to that of the subspace iteration and of the determinant search method. The absolute values of $\alpha_j^{(k)}$ calculated in the above numerical example range from 0.85 to 1.5.

Table 2 Solution Time(CPU Time, sec) of Plane Frame

Methods	Solution Time(ratio)
Proposed Method	58.1(1.00)
Subspace Iteration Method	155.4(2.7)
Determinant Search Method	133.5(2.3)

Fig. 3 Convergence of the 13th... Fig. 4 Convergence of the 14th...

Three Dimensional Building Frame

Fig. 5 Three Dimensional Building Frame

Three dimensional building frame shown in Fig. 5 consists of 191 beam elements, 100 nodes and 468 degrees-of-freedom. The mean half bandwidths of both the stiffness matrix and mass matrix are 91. $\alpha_j^{(k)}$ is applied to the 12th, the 14th and the 15th eigenpair with error norm exceeding

10^{-1}. Solution times for three methods are summarized in Table 3. If we let the solution time for the proposed method be 1, it takes 3.3 times for the subspace iteration method, 5.1 times for the determinant search method. For each solution method, the convergence of eigenpairs to which $\alpha_j^{(k)}$ is applied is presented in Figs. 6 to 7. We can see that the convergence of the proposed method is superior to that of the subspace iteration and of determinant search method. The absolute value of $\alpha_j^{(k)}$ calculated in the above numerical example has the value of 0.85 to 1.02.

Table 3 Solution Time(CPU Time, sec) of Three Dimensional Building Frame

Methods	Solution Time(ratio)
Proposed Method	217.4(1.00)
Subspace Iteration Method	723.9(3.3)
Determinant Search Method	1111.4(5.1)

Fig. 6 Convergence of the 13th... Fig. 7 Convergence of the 14th...

4. CONCLUSIONS

The paper proposes an efficient solution method using the accelerated Newton-Raphson method for eigenproblems. As shown in numerical examples of section 3, characteristics of the proposed method are identified as follows;

- Since each eigenpair is obtained independently, an eigenpair is not affected by eigenpairs previously calculated.
- The proposed method is simple and numerically stable, and converges very fast.
- Missed eigenpairs can be detected with negligible operations in passing and can be found by

the proposed method.

5. ACKNOWLEDGMENT

The research was partially supported by the Korea Science and Engineering Foundation(No: 941-1200-022-2). The support of the Korea Science and Engineering Foundation is greatly appreciated.

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Figure Lists

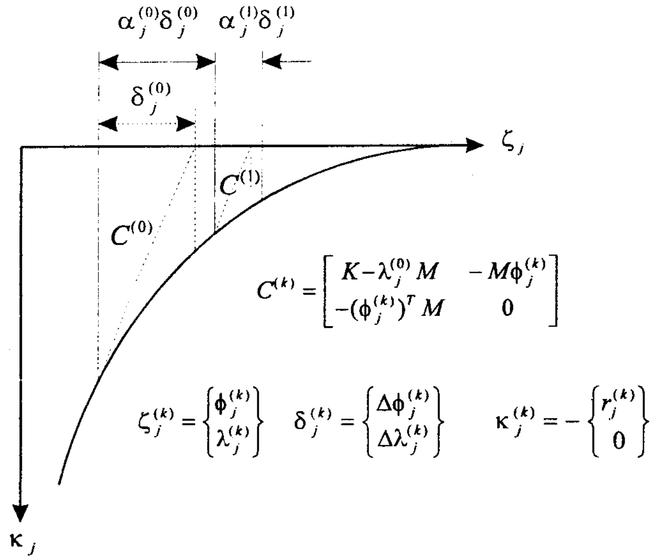
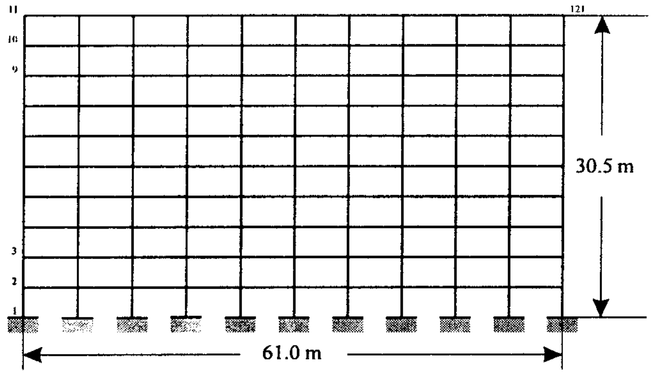


Fig. 1 The Accelerated Newton-Raphson Method



$A = 0.2787\ m^2 \qquad I = 8.631 \times 10^{-3}\ m^4$

$E = 2.068 \times 10^{10}\ Pa \qquad \rho = 5.154 \times 10^2\ kg/m^3$

Fig. 2 Plane Frame Structure

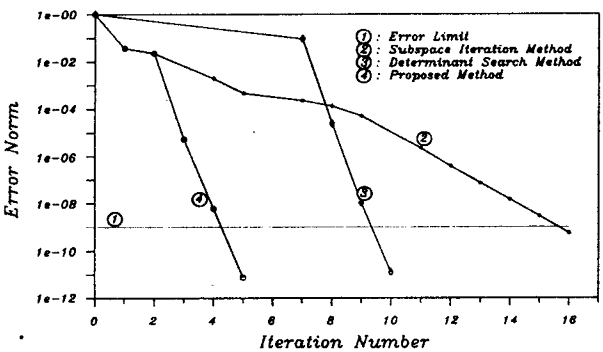


Fig. 3 Convergence of the 13th Eigenpair of Plane Frame Structure

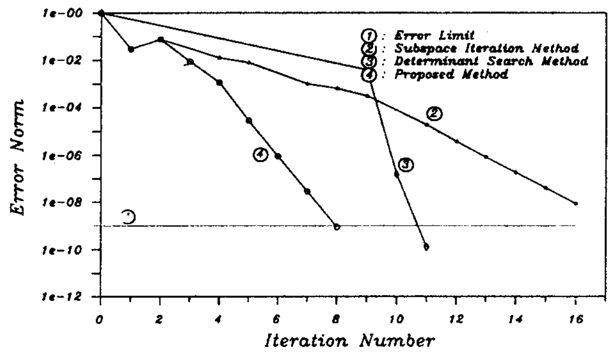
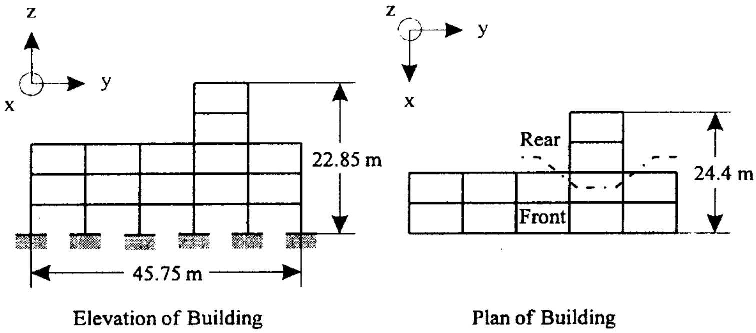


Fig. 4 Convergence of the 14th Eigenpair of Plane Frame Structure



Column in Front Building : $A = 0.2787\ m^2,\ I = 8.631\times10^{-3}\ m^4$

Column in Rear Building : $A = 0.3716\ m^2,\ I = 10.789\times10^{-3}\ m^4$

All Beams into x - Direction : $A = 0.6096\ m^2,\ I = 6.473\times10^{-3}\ m^4$

All Beams into y - Direction : $A = 0.2787\ m^2,\ I = 8.631\times10^{-3}\ m^4$

$E = 2.068\times10^{10}\ Pa,\ \rho = 5.154\times10^2\ kg/m^3$

Fig. 5 Three Dimensional Building Frame

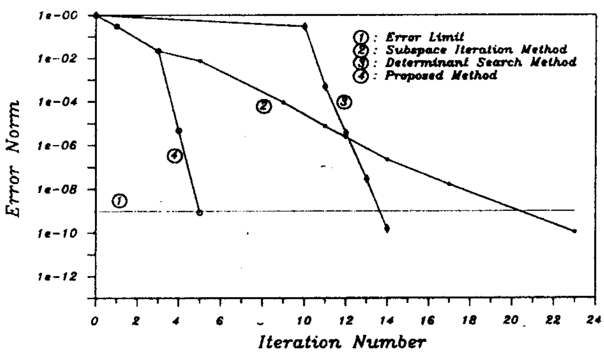


Fig. 6 Convergence of the 13th Eigenpair of 3-D. Building Structure

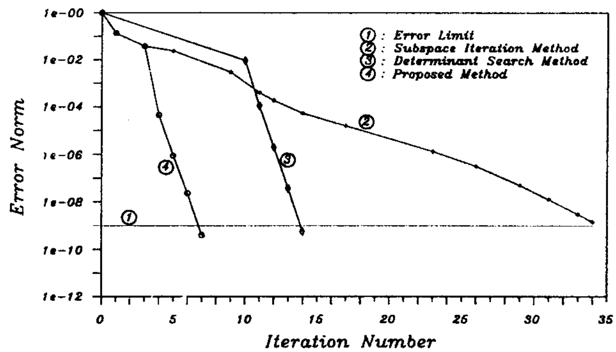


Fig. 7 Convergence of the 14th Eigenpair of 3-D. Building Structure