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An Efficient Solution Method of Quadratic Eigenproblems

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ABSTRACT

An efficient solution method is described to solve the quadratic eigenproblem arising in the dynamic analysis of general structural systems. The modified Newton-Raphson technique is employed to solve the linear eigenproblem which is derived by formulating the quadratic system of equation to a linear one by doubling the order of the system. A test problem is used to assess the performance of the proposed method for generalized eigenproblems. CPU time spent on calculating eigenpairs and convergence are compared with those of the subspace iteration method extended to the complex eigenproblem by Leung, and the results are very good.

INTRODUCTION

To determine the free or forced vibration of large dynamic systems, the efficient numerical analysis first requires that an eigenanalysis be performed. The process is low in cost, and straightforward if the damping is proportional, or more generally, if the mass, damping and stiffness matrices satisfy a condition developed by Caughey and O'Kelly(1965). If the system is nonclassical, the eigenanalysis becomes relatively expensive because it is conventionally performed in a space of twice the system's dimension, and because complex arithmetic is required.

The high relative cost of the eigenanalysis of nonclassically damped systems is possibly a motivation for investigators who explored means for analysis of such systems. Transformation methods such as QR(Moler and Stewart 1973), LZ(Kaufman 1974) or Jacobi(Veselic 1983) determine all the eigenvalues and the associated eigenvectors. This is not very efficient in situations where only the lowest frequencies are of interest and there is a large number of degrees of freedom. Also transformation methods by their nature modify the initial matrices during the

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solution process and can not take full advantage of the sparseness of these matrices.

The unsymmetric(Kim and Craig 1988), symmetric Lanczos method(Chen and Taylor 1988) or Arnoldi's method(Chen 1994) can also be used to perform the eigenanalysis. Although real arithmetic is used during the solution process, the level of the accuracy of the solutions obtained is low.

Gupta(1981) has proposed a solution procedure based on a combined Sturm sequence and inverse iteration technique. In this procedure, the calculation of each eigenvalue requires several factorizations of the matrix $A-\lambda B$ for various trial values of λ. So that it becomes less attractive for systems with a large bandwidth.

Leung(1995) has extended the subspace iteration method for real symmetric eigenproblem to the complex eigenproblem, which is a more efficient alternative than the inverse iteration method. However, as in the inverse iteration method, a large number of complex arithmetic are required in the iteration process, and singularity occurs in triangularization process when a shift value is close to the eigenvalue λ.

In this paper, the efficient method for solving a large eigenproblem is developed. In the second section, the basic concept of the proposed method which applies the modified Newton-Raphson technique to a eigenproblem is presented. In the third section, a numerical example is presented to identify the efficiency of the proposed method. Concluding remarks are made in section 4.

METHOD OF ANALYSIS

Problem Definition

The equations for free vibration of the n th order dynamic system may be written as

$$M\ddot{x}+C\dot{x}+Kx=0, \tag{1}$$

where M, C and K are the $(n\times n)$ mass, damping and stiffness matrices, respectively, and x is the $(n\times 1)$ vector of system displacements.

For symmetric and positive definite matrices, the eigenvalues and the associated eigenvectors of the system described in Eq. (1) may be determined in a straightforward, efficient manner provided that the damping matrix is proportional or, more generally(Caughey and O'Kelly 1965), provided that

$$CM^{-1}K=KM^{-1}C. \tag{2}$$

Systems for which Eq. (2) is not satisfied are called the nonclassically damped. The eigenanalysis for such systems is traditionally performed in the space extended $2n$-dimension such as

$$\begin{bmatrix}-K & 0\\ 0 & M\end{bmatrix}\begin{Bmatrix}\phi\\ \lambda\phi\end{Bmatrix}=\lambda\begin{bmatrix}C & M\\ M & 0\end{bmatrix}\begin{Bmatrix}\phi\\ \lambda\phi\end{Bmatrix}, \tag{3}$$

where λ and ϕ are the eigenvalue and the associated eigenvector of the system, respectively. Eq. (3) may be written as

$$Az=\lambda Bz \tag{4}$$

with

$$A=\begin{bmatrix}-K & 0\\ 0 & M\end{bmatrix},\quad B=\begin{bmatrix}C & M\\ M & 0\end{bmatrix}\quad\text{and}\quad z=\begin{Bmatrix}\phi\\ \lambda\phi\end{Bmatrix} \tag{5}$$

Since both the matrices A and B are not positive definite although are symmetric, the

eigenvalues and the associated eigenvectors are complex values in general.

In the following section, the proposed method with guaranteed numerical stability is described.

Proposed Eigenanalysis

Suppose that initial approximate solutions $\lambda_j^{(0)}$ and $z_j^{(0)}$ of the j th eigenvalue and the associated eigenvector of Eq. (4) are known, where the superscript denotes the iteration number. Initial values can be obtained as the intermediate results of iteration methods(Chen and Taylor 1986, Leung 1995) or final ones of approximate methods(Chen and Taylor 1988). Denoting the approximate eigenvalue and the associated eigenvector after k iterations by $\lambda_j^{(k)}$ and $z_j^{(k)}$ yields

$$r_j^{(k)} = A z_j^{(k)} - \lambda_j^{(k)} B z_j^{(k)}, \tag{6}$$

where $z_j^{(k)}$ satisfies the orthonormal condition with respect to B matrix such as

$$\left(z_j^{(k)}\right)^T B z_j^{(k)} = 1. \tag{7}$$

The residual vector $r_j^{(k)}$ is not generally zero because the approximate eigenpairs are substituted into Eq. (4). In order to get a solution converged to the eigenvalue and the associated eigenvector of the system, the residual vector should be removed. Apply the Newton-Raphson technique for this purpose to the eigenproblem as follows;

$$\begin{aligned} r_j^{(k+1)} &= A z_j^{(k+1)} - \lambda_j^{(k+1)} B z_j^{(k+1)} \\ &= 0 \end{aligned}, \tag{8}$$

where $z_j^{(k+1)}$ also satisfies the orthonormal condition with respect to B matrix such as

$$\left(z_j^{(k+1)}\right)^T B z_j^{(k+1)} = 1 \tag{9}$$

and then $$\lambda_j^{(k+1)} = \lambda_j^{(k)} + \Delta\lambda_j^{(k)} \quad \text{and} \quad z_j^{(k+1)} = z_j^{(k)} + \Delta z_j^{(k)}. \tag{10, 11}$$

Upon substituting Eqs. (6), (7), (10) and (11) into Eqs. (8), (9) and neglecting their higher order terms $\Delta\lambda_j^{(k)} B \Delta z_j^{(k)}$ and $\left(\Delta z_j^{(k)}\right)^T B \Delta z_j^{(k)}$ yield

$$(A - \lambda_j^{(k)} B) \Delta z_j^{(k)} - \Delta\lambda_j^{(k)} B z_j^{(k)} = - r_j^{(k)} \tag{12}$$

and $$\left(z_j^{(k)}\right)^T B \Delta z_j^{(k)} = 0, \tag{13}$$

where $\Delta\lambda_j^{(k)}$ and $\Delta z_j^{(k)}$ are unknown incremental values corresponded to $\lambda_j^{(k)}$ and $z_j^{(k)}$

Writing Eqs. (12) and (13) in matrix form yields

$$\begin{bmatrix} A - \lambda_j^{(k)} B & - B z_j^{(k)} \\ -\left(B z_j^{(k)}\right)^T & 0 \end{bmatrix} \begin{Bmatrix} \Delta z_j^{(k)} \\ \Delta\lambda_j^{(k)} \end{Bmatrix} = - \begin{Bmatrix} r_j^{(k)} \\ 0 \end{Bmatrix} \tag{14}$$

The symmetric coefficient matrix of the above linear algebraic equation is of order $(2n+1)$. If all eigenvalues are distinct, $\lambda_i \neq \lambda_j$ $(i \neq j)$, then the coefficient matrix is nonsingular. If the shift in the iteration methods is near to an eigenvalue, numerical instability may be encountered. In proposed method, the numerical stability problem, however, can be solved by means of including a side condition $\left(z_j^{(k)}\right)^T B \Delta z_j^{(k)} = 0$ as shown in algebraic Eq. (14). This is the main

difference compared with the classically inverse iteration method with shift. The proposed method used the Newton-Raphson technique, despite of its rapid convergence, is not efficient, since the new coefficient matrix has to be reformed and refactorized in each iteration step.

The complicated elimination procedure in each iteration may be overcome by applying the modified Newton-Raphson technique, that is,

$$\begin{bmatrix} A-\lambda_j^{(0)}B & -Bz_j^{(k)} \\ -\left(Bz_j^{(k)}\right)^T & 0 \end{bmatrix} \begin{Bmatrix} \Delta z_j^{(k)} \\ \Delta\lambda_j^{(k)} \end{Bmatrix} = -\begin{Bmatrix} r_j^{(k)} \\ 0 \end{Bmatrix} \tag{15}$$

The symmetric coefficient matrix of Eq. (15) is nonsingular. Once the submatrix $A-\lambda_j^{(0)}B$ is decomposed into LDL^T (L : lower triangular matrix, D : diagonal matrix), a small number of operations are required to solve Eq. (15), since only the vector $Bz_j^{(k)}$ in the coefficient matrix is changed in each iteration.

NUMERICAL EXAMPLE

In this section, a test problem with nonclassically damping is presented to show the effectiveness of the proposed method. The symmetric Lanczos method(Chen and Taylor 1988) with partial reorthogonalization scheme is used to obtain the initial values of the proposed method. The lowest p initial eigenpairs can be obtained by solving the standard eigenproblem reduced by $2p$ Lanczos vectors. The approximate eigenpairs, then, are improved by the iterations until their error norm are within 10^{-6}. The error norm is defined as

$$error\ norm = \frac{\left\|\left(A-\lambda_j^{(k)}B\right)z_j^{(k)}\right\|_2}{\left\|Az_j^{(k)}\right\|_2} \tag{28}$$

All executions are done on the CONVEX C3420 with 100 MIPS and 200 MFLOPS.

Cantilever Beam with Multi-Lumped Dampers

The example studied is a cantilever beam with multi-lumped translational viscous-dampers attached at each node. The geometrical configuration is shown in Fig. 1. The structural model could be considered as a representative of a soil-structure interaction. The structure is modeled by 100 equal elements and has 200 degrees of freedom. The order of the associated (A,B) is 400. Dimensionless value of Young's modulus for the beam material is taken as 1000, while mass density, section area and inertia are specified to be of unit value. The mass matrix is consistent one. The damping matrix C consists of the Rayleigh damping and concentrated dampers. The Rayleigh damping is a linear combination of the stiffness and mass matrices such as

$$C = \alpha M + \beta K, \tag{29}$$

where the Rayleigh coefficients α and β are 0.001, respectively. On the other hand, the damping coefficients of the concentrated dampers are 0.1.

The results for the lowest five eigenpairs with error norm 10^{-6} are summarized in Table 1. The initial values of the proposed method which are calculated by the Lanczos method are about 10^{-1} to 10^{-3}. Only one iteration is needed to obtain the eigenpairs with error norm 10^{-6}. The CPU time for the proposed method is compared with that of the subspace iteration method

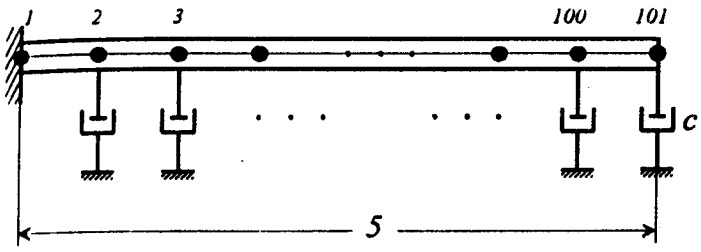

Fig.1 *Cantilever beam with multi-lumped dampers*

in Table 2. If we let the solution time for the proposed method be 1, it takes 1.25 times for the subspace iteration method. The variations of the error norms for the each eigenpair corresponding to the iteration number are depicted in Figs. 2 to 6. The first step of the proposed method denotes the result of the Lanczos algorithm. The results indicate that the convergence of the proposed method is much better than that of the subspace iteration method.

Table 1 *The results of the proposed method(Number of iterations, eigenvalue and error norm)*

Mode number	Error norm of starting value (Lanczos method)	Proposed method		
		Number of iterations	Eigenvalue	Error norm
1	0.872989E-04	1	-1.02232 ± i 3.95028	0.183316E-07
2	0.763146E-03	1	-1.18011 ± i 18.3991	0.189217E-09
3	0.437867E-04	1	-1.79640 ± i 39.6535	0.373318E-10
4	0.605684E-02	1	-2.87171 ± i 60.9945	0.371279E-11
5	0.420530E-00	1	-4.40255 ± i 82.2930	0.983166E-07

Table 2 *CPU time spent for first five eigenpairs*

Method	CPU time(in seconds)	Ratio
Subspace iteration method	96.10	1.25
Proposed method	76.75	1.00
(Lanczos method + Iteration scheme)	(10.55 + 66.20)	

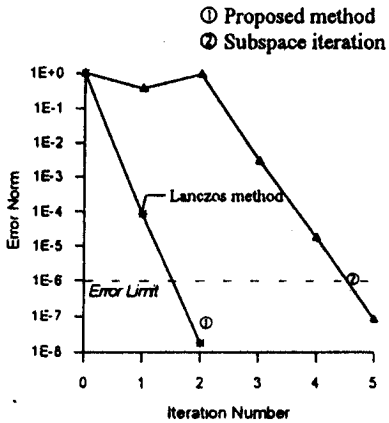

Fig.2 *Error norm of the 1st Eigenpair*

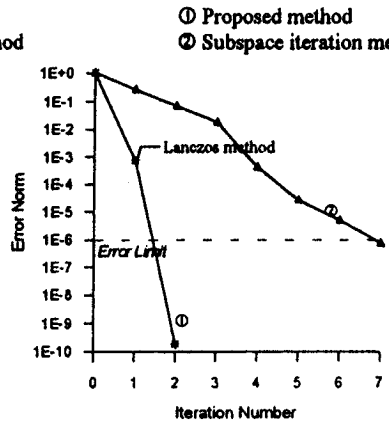

Fig.3 *Error norm of the 3rd Eigenpair*

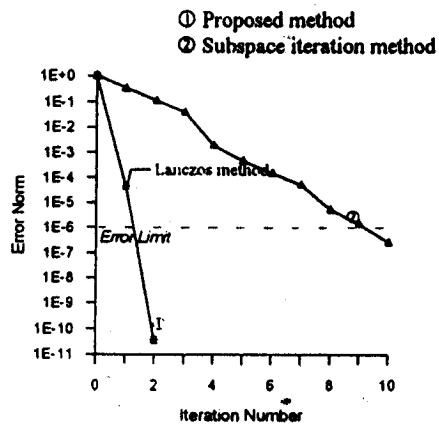

Fig.4 *Error norm of the 5th Eigenpair*

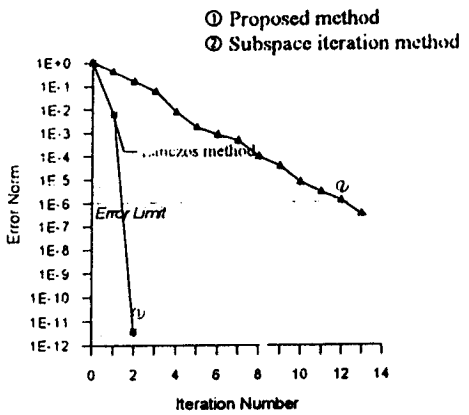

Fig.5 *Error norm of the 7th Eigenpair*

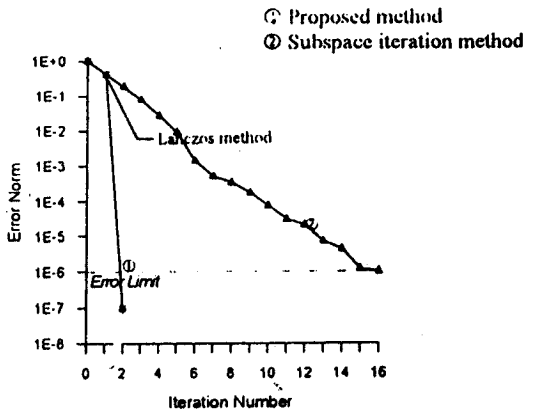

Fig.6 *Error norm of the 9th Eigenpair*

CONCLUSIONS

In this paper, The method for finding the eigenpairs of the nonclassically damped systems is proposed. As shown in numerical analysis, section 3, characteristics of the proposed method are identified as follows;

. ① The method takes full advantage of the sparseness of the system matrices.

② The proposed method guarantees the numerical stability, which is proved analytically.

③ The proposed method has high convergence rate.

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