Natural Frequency and Mode Shape Sensitivities of Damped Systems : Part I, Distinct Natural Frequencies

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ABSTRACT

An analytical procedure is presented for determining the sensitivities of eigenvalues and eigenvectors for the proportionally and non-proportionally damped vibratory systems with distinct eigenvalues. The eigenpair derivatives of the structural and mechanical damped system can be obtained consistently by solving algebraic equations with symmetric matrix. The algorithm of the method is very simple and compact and the method can find the exact solutions. Furthemore, its numerical stability is proved. As an example of a structural system to illustrate the theory and its possibilities in the case of proportionally damped system, a cantilever plate is considered, and 7-DOF half-car model as a mechanical system in the case of non-proportionally damped system.

INTRODUCTION

The dynamic responses of the structural or mechanical systems can be completely identified by obtaining the natural frequencies and mode shapes of systems. Variations in system parameters lead to changes in these dynamic characteristics and hence in responses. The derivatives of the eigenpairs are useful in design trend studies and for gaining insight into the behavior of physical systems. The cost of reanalyzes can be remarkably reduced by using these eigenpair derivatives in large systems. The derivatives of the mode shapes with respect to design parameters are particularly useful in certain analysis and design applications;

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approximating a new vibration mode shape due to a perturbation in a design parameter, determining the effect of design changes on the dynamic behavior of systems[1], tailoring mode shapes to minimize displacements at certain points on a system[2]. In contrast to computing eigenvalue derivatives where prefered methods exist, there are a number of different methods for calculating mode shape derivatives. The different methods seek to overcome the practical difficulty of solving a singular matrix equation; finite-difference method[3], iterative method[4-5], modal method[6-8], Nelson's method[9] and Lee and Jung's method[10-11], etc.

A number of the prescribed methods can be applied to the damped system; Pomazal and Snyder[12] extended the theory to the complex eigenvalue problem to analyze the effects of adding springs and dampers to viscously damped systems. Hallquist[13] presented for determining the effects of mass modification in viscously damped systems. Recently Zimoch[14] presented the sensitivity analysis method for determining the dynamic characteristics of mechanical systems to variations in the parameters. The method is applied to conservative as well as non-conservative systems, however may be restricted to the mechanical systems (lumped systems) with only distinct natural frequencies; it has some difficulties in applying to the structural system which has the structural parameters.

In this paper, the algebraic method for calculating the eigenpair derivatives worked by Lee and Jung[10] is extended for the proportionally and non-proportionally damped systems with distinct eigenvalues.The proposed method can find the eigenvector derivatives of the structural and mechanical damped system by solving the algebraic equations with symmetric matrix which is added side condition; The proposed algorithm can be applied consistently to the structural and lumped mechanical systems. Note that the proposed method is powerful when the multipled eigenvalues exist(refer to Part II).

SENSITIVITY ANALYSIS OF DAMPED SYSTEMS

Technical background

The eigenvalue problem of a damped system can be expressed as

$$(\lambda_j^2 M+\lambda_j C+K)\ \phi_j=0\ . \tag{1}$$

where M is positive definite and K positive definite or semi-positive definite. Let the orthogonality condition be the following equation when all eigenvalues are distinct,

$$\phi_j^T\ M\ \phi_j=1\ . \tag{2}$$

To obtain the eigenvalue derivative, Eq.(1) is differentiated with respect to design parameter p,

$$(\lambda_j^2 M+\lambda_j C+K)\ \phi_j' \ =-(2\lambda_j M+C)\ \phi_j\ \lambda_j'-\left(\lambda_j^2\ M'+\lambda_j\ C'+K'\right)\phi_j\ . \tag{3}$$

Premultiplying by ϕ_j^T at each side of Eq.(3), the eigenvalue derivative can be obtained as

$$\lambda_j'=-\ \phi_j^T\left(\lambda_j^2 M'+\lambda_j\ C'+K'\right)\phi_j\Big/\phi_j^T\left(2\lambda_j M+C\right)\phi_j\ \ , \tag{4}$$

hence the right hand side of Eq.(3) is known but eigenvector derivative can not be found directly since the matrix $[\lambda_j^2\ M+\lambda_j\ C+K]$ is singular. To find the eigenvector derivative in the above linear algebraic equation, the algebraic method worked by Lee and Jung[10] is extended for the proportionally and non-proportionally damped systems with distinct eigenvalues.

Proposed sensitivity analysis of damped system

The proposed method solves a symmetric linear algebraic equation with side condition. Differentiating the normalization condition Eq.(2) with respect to the design parameter, then

$$\phi_j^T M \phi_j' + 0.5 \phi_j^T M' \phi_j = 0 \ . \tag{5}$$

Eq.(3) and Eq.(5) may be written as the single matrix equation as

$$A^* \begin{Bmatrix} \phi_j' \\ 0 \end{Bmatrix} = f_j \ , \tag{6}$$

where

$$A^* = \begin{bmatrix} \lambda_j^2 M + \lambda_j C + K & -M\phi_j \\ -\phi_j^T M & 0 \end{bmatrix} \text{ and } f_j = \begin{Bmatrix} -(2\lambda_j M + C)\phi_j \lambda_j' - \left(\lambda_j^2 M' + \lambda_j C' + K'\right)\phi_j \\ 0.5\,\phi_j^T M' \phi_j \end{Bmatrix}. \tag{7, 8}$$

The coefficient matrix can be decomposed into upper and lower triangular forms than a forward and backward substitution scheme may be used to evaluate the components of ϕ_j'.

NUMERICAL STABILITY OF THE PROPOSED METHOD

Let's define the matrix A^* to prove the nonsingularity for the jth eigenpair,

$$A^* = \begin{bmatrix} \lambda_j^2 M + \lambda_j C + K & -M\phi_j \\ -\phi_j^T M & 0 \end{bmatrix} . \tag{9}$$

Introducing the following new eigenvalue problem to prove the nonsingularity of A^*,

$$A^* u_i = \gamma_i B^* u_i \qquad i = 1, 2, \cdots, n+1 , \tag{10}$$

where

$$B^* = \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix} . \tag{11}$$

Eq. (10) can be written collectively as

$$A^* U = B^* U \Gamma \ , \tag{12}$$

where $\quad U = [u_1 \ u_2 \cdots u_{n+1}] \quad$ and $\quad \Gamma = diag(\gamma_1 \ \gamma_2 \cdots \gamma_{n+1}) \ .$ (13, 14)

The eigenvectors and eigenvalues of the new eigenvalue problem Eq.(12) used the jth eigenpair are as follows;

- eigenvectors, $u's$: $\dfrac{1}{\sqrt{2}}\begin{Bmatrix} \phi_j \\ 1 \end{Bmatrix}, \ \dfrac{1}{\sqrt{2}}\begin{Bmatrix} \phi_j \\ -1 \end{Bmatrix}, \ \begin{Bmatrix} \phi_k \\ 0 \end{Bmatrix} \qquad k = 1, 2, \cdots, n, \quad k \neq j$ (15)
- eigenvalues, $\gamma's$: $-1, \ 1, \ (\lambda_j - \lambda_k)(\lambda_j - \bar{\lambda}_k) \qquad k = 1, 2, \cdots, n, \quad k \neq j$ (16)

Note that the above eigenvectors satisfies the B^*-normalization condition;

$$u_i^T A^* u_j = \gamma_i \delta_{ij} \quad \text{and} \quad u_i^T B^* u_j = \delta_{ij} \ , \tag{17, 18}$$

where δ_{ij} is Kronecker delta. Considering the determinant of Eq.(12),

$$\det[A^*] = \det[B^*] \ \det[\Gamma] \ = -\det[M] \prod_{\substack{k=1 \\ k \neq j}}^{n} (\lambda_j - \lambda_k)(\lambda_j - \bar{\lambda}_k) . \tag{19}$$

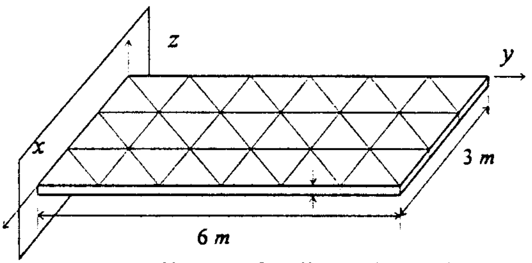
Suppose that all eigenvalues are different, the determinant of A^* is not equal to zero because the determinant of M is non-zero by definition. The numerical stability of the proposed method in the case of distinct natural frequencies is proved.

NUMERICAL EXAMPLES

To demonstrate the theory of the proposed method and its possibilities, two numerical examples are presented. In the first, as an example of a damped system with proportional damping, the finite element model of the cantilever plate is considered, while the second a half-car modeled 7-DOF is considered to demonstrate an application of the proposed method to non-proportionally damped system with distinct natural frequencies.

Cantilever plate (proportionally damped system)

The finite element model of the cantilever plate used in Reference [5] is modeled with 36 triangular elements as shown in Figure 1. Each node of the element has three degrees of freedom(z-translation, x-rotation and y-rotation), hence the total degrees of freedom of the structure 72. For example calculations, Young's modulus is $10.5 \times 10^5\ N/m^2$, the mass density $5.88 \times 10^{-3}\ kg/m^2$. The length of the plate is 6 m, and width 3 m and thickness 0.01 m.



Number of nodes : 28
Number of elements : 36
Number of degrees of freedom : 72
Young's modulus : $E = 10.5 \times 10^5\ N/m^2$
Mass density : $\rho = 5.88 \times 10^{-3}\ kg/m^3$

Figure 1. Cantilever plate with the thickness t as the design parameter.

Assume that damping matrix is a linear combination of the stiffness and mass matrices as

$$C = \alpha K + \beta M \tag{20}$$

where α and β are the Rayleigh coefficients. The design parameter is the plate thickness t. The stiffness and mass matrices of the structure are proportional to t^3 and t, respectively. The derivatives of the stiffness and mass matrices can be immediately obtained by differentiating them with respect to t, and the derivative of the damping matrix by combining them.

Some sensitivity results are represented in Tables 1 to 2. The lowest ten frequencies and their derivatives of the cantilever plate are listed in Table 1. Some components of the actual and approximated eigenvectors according to the various Δt are represented in Table 2. The eigenvectors calculated by the sensitivity analysis must be normalized by the mass matrix changed at the end of the sensitivity analysis.

Table 1. *The lowest ten eigenvalues and their derivatives with respect to the thickness t.*

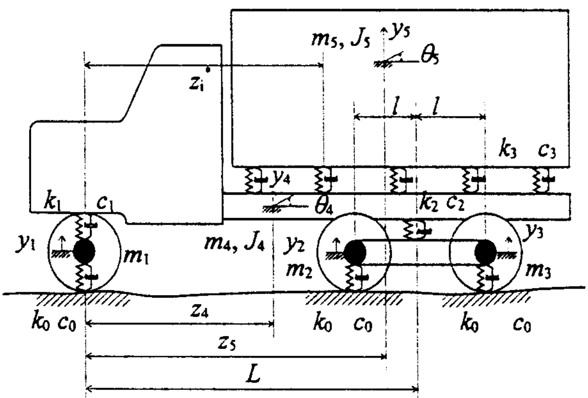
Eigenvalue number	Eigenvalues	Eigenvalue derivatives
1, 2	$-0.0015367 \mp j5.4528433$	$-0.2973350 \mp j545.28429$
3, 4	$-0.0236708 \mp j21.735135$	$-4.7241664 \mp j2173.5109$
5, 6	$-0.0548020 \mp j33.091330$	$-10.950391 \mp j3309.1239$
7, 8	$-0.2344999 \mp j68.475851$	$-46.889971 \mp j6847.5048$
9, 10	$-0.4268189 \mp j92.386121$	$-85.353776 \mp j9238.4150$

Table 2. *The exact and approximated eigenvectors after the change of thickness* $\Delta t/t = 0.1$.

Equation Number	First eigenvector		Third eigenvector	
	Exact	Approx.	Exact	Approx.
1	−3.28614541	−3.28614605	11.5026018	11.5026022
2	−6.67026878	−6.67026981	19.4853397	19.4853370
3	1.21839299	1.21839300	6.96765808	6.96765833
4	−3.80906375	−3.80906436	3.09419092	3.09419063
⋮	⋮	⋮	⋮	⋮
69	−0.32948144	−0.32948110	71.8098435	71.8098375
70	−83.1166213	−83.1166189	−108.367441	−108.367434
71	−19.9142313	−19.9142282	−8.76763697	−8.76763254
72	−0.54454783	−0.54465730	70.9140661	70.9140577

7-DOF half-car model (non-proportionally damped system)

The second example problem for the non-proportionally damped system is used in reference [14] and shown in Figure 2. The truck is modeled as the lumped system with 7-DOF. Only the vibrations in vertical plane are considered.



$z_1^*=1.8$ *m*, $z_2^*=2.3$ *m*, $z_3^*=3.8$ *m*, $z_4^*=4.3$ *m*, $z_5^*=5.0$ *m*, $z_4=2.0$ *m*, $z_5=3.0$ *m*;

$m_1=75$ *kg*, $m_2=m_3=80$ *kg*, $m_4=3500$ *kg*, $m_5=1800$ *kg*; $c_0=120$ *Ns/m*, $c_1=150$ *Ns/m*, $c_2=50$ *Ns/m*, $c_3=80$ *Ns/m*; $k_0=12\ 000$ *N/m*, $k_1=14\ 000$ *N/m*, $k_2=9500$ *N/m*, $k_3=4000$ *N/m*; $L=3.5$ *m*, $l=0.85$ *m*;

Figure 2. The 7-DOF half-car model as non-proportionally damped system.

Sensitivity results are summarized in Tabels 3. The components of the exact and approximated eigenvector after the changes of a mass, a spring and a damper are represented.

The theory of the proposed method and its possibilities are demonstrated through all examples. The proposed method can find efficiently the sensitivities of the damped systems.

Table 3. *The exact and approximated eigenvectors after changing the mass, spring and damper* ($\Delta m_4 = 17.5 kg$, $\Delta c_2 = 0.25\ Ns/m$ *and* $\Delta k_3 = 20\ N/m$).

DOF Number	First eigenvector		Third eigenvector	
	Exact	Approx.	Exact	Approx.
1	−0.0004976 − j0.0000073	−0.0004962 − j0.0000072	0.0127229 − j0.0000171	0.0127233 − j0.0000170
2	−0.0044508 − j0.0000125	−0.0044511 − j0.0000125	−0.0006105 − j0.0000152	−0.0006096 − j0.0000151
3	−0.0044508 − j0.0000126	−0.0044511 − j0.0000126	−0.0006105 − j0.0000155	−0.0006096 − j0.0000154
4	−0.0092860 + j0.0000091	−0.0092854 + j0.0000091	0.0088102 − j0.0000299	0.0088123 − j0.0000298
5	−0.0041834 + j0.0000115	−0.0041843 + j0.0000114	−0.0072837 − j0.0000067	−0.0072830 − j0.0000066
6	−0.0163027 − j0.0000318	−0.0163015 − j0.0000317	0.0033087 − j0.0000033	0.0033116 − j0.0000032
7	−0.0036857 + j0.0000224	−0.0036879 + j0.0000223	−0.0088608 − j0.0000400	−0.0088585 − j0.0000399

CONCLUSIONS

This paper proposes an efficient numerical method whose stability is proved for calculation of the derivatives of vibration natural frequency and the corresponding mode shape of the structural and mechanical damped system with distinct natural frequencies; the method can be applied to the mechanical system with lumped parameters as well as the structural system with structural design parameters. The algorithm of the method can be added easily to the commercial FEM code because it finds the exact solutions and its numerical stability is proved. The method has the desirable properties of preserving the band and symmetry of the matrices, and of requiring knowledge of only one eigenpair to be differentiated.

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