# AN EFFICIENT SOLUTION METHOD OF GENERALIZED EIGENPROBLEMS USING THE MODIFIED NEWTON-RAPHSON TECHNIQUE

**Man-Cheol Kim[1], Ju-Won Oh[2], Gyu-Won Lee[3], In-Won Lee[4]**

[1]Graduate Student, Dept. of Civil Engineeering, KAIST, Taejon 305-701,KOREA
[2]Professor, Dept. of Civil Engineering, Han-Nam University, Taejon 305-701,KOREA
[3]Professor, Dept. of Civil Engineering, Chon-Buk Nat. University, Chonju 560-756,KOREA
[4]Professor, Dept. of Civil Engineering, KAIST Taejon 305-701, KOREA

An efficient solution method is described to solve the quadratic eigenproblem with distinct eigenvalues arising in the dynamic analysis of general structural systems. The modified Newton-Raphson technique is employed to solve the linear eigenproblem which is derived by reformulating the quadratic system of equation to a linear one by doubling the order of the system. Some test problems are used to assess the performance of the proposed method for generalized eigenproblems. One uses the final results of the symmetric Lanczos method with partial reorthogonalization scheme proposed by H. C. Chen as initial values of the method. CPU time spent on calculating eigenpairs and convergence are compared with those of the subspace iteration method extended to the complex eigenproblem by A. Y. T. Leung, and the results are very good.

## NOMENCLATURE

$A = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}$

$B = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}$

$C$ = Damping matrix

$F = \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}$

$K$ = Stiffness matrix

$M$ = Mass matrix

$r$ = Residual vector

$x$ = Displacement vector

$w$ = Eigenvector

$z = \begin{Bmatrix} w \\ \lambda w \end{Bmatrix}$

$\alpha, \beta$ = Coefficients of Rayleigh damping

$\lambda$ = Eigenvalue

$\Delta\lambda$ = Incremental unknown values of eigenvalue

$\Delta z$ = Incremental unknown values

*Superscript*

$(k)$ = $k$ iterations

*Subscript*

$j$ = $j$th eigenvalue or eigenvector

## INTRODUCTION

To determine the free or forced vibration of large dynamic systems, the efficient numerical analysis first requires that an eigenanalysis be performed. The process is low in cost, and straightforward if the damping is proportional, or more generally, if the mass, damping

and stiffness matrices satisfy a condition developed by Caughey and O'Kelly[1]. If the system is nonclassical, the eigenanalysis becomes relatively expensive because it is conventionally performed in a space of twice the system's dimension, and because complex arithmetic is required.

The high relative cost of the eigenanalysis of nonclassically damped systems is possibly a motivation for investigators who explored means for analysis of such systems. Transformation methods such as QR[11], LZ[8] or Jacobi[15] determine all the eigenvalues and the associated eigenvectors. This is not very efficient in situations where only the lowest frequencies are of interest and there is a large number of degrees of freedom. Also transformation methods by their nature modify the initial matrices during the solution process and can not take full advantage of the sparseness of these matrices.

The unsymmetric[9], symmetric Lanczos method[5] or Arnoldi's method[2] can also be used to perform the eigenanalysis. Although real arithmetic is used during the solution process, the level of the accuracy of the solutions obtained is low.

Gupta[6,7] has proposed a solution procedure based on a combined Sturm sequence and inverse iteration technique. In this procedure, the calculation of each eigenvalue requires several factorizations of the matrix $A - \lambda B$ for various trial values of $\lambda$. So that it becomes less attractive for systems with a large bandwidth.

Leung[10] has extended the subspace iteration method for real symmetric eigenproblem to the complex eigenproblem, which is a more efficient alternative than the inverse iteration method. However, as in the inverse iteration method, a large number of complex arithmetic are required in the iteration process, and singularity occurs in triangularization process when a shift value is close to the eigenvalue $\lambda$.

In this paper, the efficient method for solving a large eigenproblem is developed. In the first section, the basic concept of the proposed method which applies the modified Newton-Raphson technique to a eigenproblem is presented. In the second section, the numerical stability of the proposed method is established. In the third section, some numerical examples are presented to identify the efficiency of the proposed method. Concluding remarks are made in section 4.

## METHOD OF ANALYSIS

### Introduction

The equations for free vibration of the $n$ th order dynamic system may be written as

$$M\ddot{x} + C\dot{x} + Kx = 0, \tag{1}$$

where $M$, $C$ and $K$ are the $(n \times n)$ mass, damping and stiffness matrices, respectively, and $x$ is the $(n \times 1)$ vector of system displacements.

For symmetric and positive definite matrices, the eigenvalues and the associated eigenvectors of the system described in eqn (1) may be determined in a straightforward, efficient manner provided that the damping matrix is proportional or, more generally[1], provided that

$$C M^{-1} K = K M^{-1} C. \tag{2}$$

Systems for which eqn (2) is not satisfied are called the nonclassically damped. The eigenanalysis for such systems is traditionally performed in the space extended $2n$-

dimension such as

$$\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} w \\ \lambda w \end{Bmatrix} = \lambda \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} w \\ \lambda w \end{Bmatrix}, \tag{3}$$

where $\lambda$ and $w$ are the eigenvalue and the associated eigenvector of the system, respectively.
Eqn (3) may be written as

$$A z = \lambda B z \tag{4}$$

with

$$A = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \quad B = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \quad \text{and} \quad z = \begin{Bmatrix} w \\ \lambda w \end{Bmatrix}. \tag{5}$$

Since both the matrices $A$ and $B$ are not positive definite although are symmetric, the eigenvalues and the associated eigenvectors are complex values in general.

In the following section, the proposed method with guaranteed numerical stability is described to find the eigenpairs.

## Proposed Eigenanalysis

Suppose that initial approximate solutions $\lambda_j^{(0)}$ and $z_j^{(0)}$ of the $j$th eigenvalue and the associated eigenvector of eqn (4) are known, where the superscript denotes the iteration number. Initial values can be obtained as the intermediate results of iteration methods[4,6,7,8,10,11,15] or final ones of approximate methods[2,3,5,9,13]. In this paper, the initial values are taken as the final results of the symmetric Lanczos method[5] with partial reorthogonalization scheme[12,14] proposed by H. C. Chen. Denoting the approximate eigenvalue and the associated eigenvector after $k$ iterations by $\lambda_j^{(k)}$ and $z_j^{(k)}$ yields

$$r_j^{(k)} = A z_j^{(k)} - \lambda_j^{(k)} B z_j^{(k)}, \tag{6}$$

where $z_j^{(k)}$ satisfies the orthonormal condition with respect to $B$ matrix such as

$$\left(z_j^{(k)}\right)^T B z_j^{(k)} = 1. \tag{7}$$

The residual vector $r_j^{(k)}$ is not generally zero because the approximate eigenpairs are substituted into eqn (4). In order to get a solution converged to the eigenvalue and the associated eigenvector of the system, the residual vector should be removed. Apply the Newton-Raphson technique for this purpose to the eigenproblem as follows;

$$\begin{aligned} r_j^{(k+1)} &= A z_j^{(k+1)} - \lambda_j^{(k+1)} B z_j^{(k+1)} \\ &= 0 \end{aligned}, \tag{8}$$

where $z_j^{(k+1)}$ also satisfies the orthonormal condition with respect to $B$ matrix such as

$$\left(z_j^{(k+1)}\right)^T B\, z_j^{(k+1)} = 1 \tag{9}$$

and then

$$\lambda_j^{(k+1)} = \lambda_j^{(k)} + \Delta\lambda_j^{(k)} \quad \text{and} \quad z_j^{(k+1)} = z_j^{(k)} + \Delta z_j^{(k)}. \tag{10, 11}$$

Upon substituting eqns (6), (7), (10) and (11) into eqns (8), (9) and neglecting their higher order terms $\Delta\lambda_j^{(k)}\, B\, \Delta z_j^{(k)}$ and $\left(\Delta z_j^{(k)}\right)^T B\, \Delta z_j^{(k)}$ yield

$$(A - \lambda_j^{(k)}\, B)\, \Delta z_j^{(k)} - \Delta\lambda_j^{(k)}\, B\, z_j^{(k)} = -\, r_j^{(k)} \tag{12}$$

and

$$\left(z_j^{(k)}\right)^T B\, \Delta z_j^{(k)} = 0, \tag{13}$$

where $\Delta\lambda_j^{(k)}$ and $\Delta z_j^{(k)}$ are unknown incremental values corresponded to $\lambda_j^{(k)}$ and $z_j^{(k)}$.

Writing eqns (12) and (13) in matrix form yields

$$\begin{bmatrix} A - \lambda_j^{(k)}\, B & -B\, z_j^{(k)} \\ -\left(B\, z_j^{(k)}\right)^T & 0 \end{bmatrix} \begin{Bmatrix} \Delta z_j^{(k)} \\ \Delta\lambda_j^{(k)} \end{Bmatrix} = -\begin{Bmatrix} r_j^{(k)} \\ 0 \end{Bmatrix}. \tag{14}$$

The symmetric coefficient matrix of the above linear algebraic equation is of order $(2n+1)$. If all eigenvalues are distinct, $\lambda_i \neq \lambda_j\ (i \neq j)$, then the coefficient matrix is nonsingular. Nonsingularity will be proved in next section 2.3. If the shift is near to an eigenvalue, numerical instability in the iteration methods may be encountered. In proposed method, the numerical stability problem, however, can be solved by means of including a side condition $\left(z_j^{(k)}\right)^T B\, \Delta z_j^{(k)} = 0$ as shown in algebraic eqn (14). This is the main difference compared with the classically inverse iteration method with shift. The proposed method used the Newton-Raphson technique, despite of its rapid convergence, is not efficient, since the new coefficient matrix has to be reformed and refactorized in each iteration step.

The complicated elimination procedure in each iteration may be overcome by applying the modified Newton-Raphson technique, that is,

$$\begin{bmatrix} A - \lambda_j^{(0)}\, B & -B\, z_j^{(k)} \\ -\left(B\, z_j^{(k)}\right)^T & 0 \end{bmatrix} \begin{Bmatrix} \Delta z_j^{(k)} \\ \Delta\lambda_j^{(k)} \end{Bmatrix} = -\begin{Bmatrix} r_j^{(k)} \\ 0 \end{Bmatrix}. \tag{15}$$

The symmetric coefficient matrix of eqn (15) is nonsingular(refer to numerical stability, section 2.3). Once the submatrix $A - \lambda_j^{(0)}\, B$ is decomposed into $L\, D\, L^T$($L$ : lower triangular matrix, $D$ : diagonal matrix), a small number of operations are required to solve eqn (15), since only the vector $B\, z_j^{(k)}$ in the coefficient matrix is changed in each iteration. The complete procedure of the proposed method for finding the eigenpairs is summarized in Table 1.

Table 1. *Algorithm of the Proposed Method*

1. Calculate initial eigenvalues $\lambda_j^{(0)}$ and eigenvectors $z_j^{(0)}$ for $j = 1, 2, \ldots, p$, where $p$ is the number of eigenvalues to be sought.
2. Iterate the following procedure for each eigenvalue and the associated eigenvector.
   (a) For $j = 1$
   (b) For $k = 0$
   (c) Define $\begin{bmatrix} A - \lambda_j^{(0)} B & -B z_j^{(k)} \\ -\left(B z_j^{(k)}\right)^T & 0 \end{bmatrix}$.
   (d) Compute $-\begin{Bmatrix} r_j^{(k)} \\ 0 \end{Bmatrix}$, where $r_j^{(k)} = (A - \lambda_j^{(k)} B)\, z_j^{(k)}$.
   (e) If the norm of the residual vector is reduced by the predetermined error limit, Stop
   (f) Find $\Delta\lambda_j^{(k)}$ and $\Delta z_j^{(k)}$ by solving
   $$\begin{bmatrix} A - \lambda_j^{(0)} B & -B z_j^{(k)} \\ -\left(B z_j^{(k)}\right)^T & 0 \end{bmatrix} \begin{Bmatrix} \Delta z_j^{(k)} \\ \Delta\lambda_j^{(k)} \end{Bmatrix} = \begin{Bmatrix} -r_j^{(k)} \\ 0 \end{Bmatrix}$$
   (g) Compute $\lambda_j^{(k+1)} = \lambda_j^{(k)} + \Delta\lambda_j^{(k)}$ and $z_j^{(k+1)} = z_j^{(k)} + \Delta z_j^{(k)}$.
   (h) Go to Step (b) with $k = k + 1$.

**Numerical Stability of the Proposed Method**

The most remarkable characteristic of the proposed method is that its numerical stability is guaranteed. The numerical stability can be proved by identifying the nonsingularity of the coefficient matrix of eqn (15). Let the coefficient matrix of eqn (15) be denoted by $E^{(k)}$, that is,

$$E^{(k)} = \begin{bmatrix} A - \lambda_j^{(0)} B & -B z_j^{(k)} \\ -\left(B z_j^{(k)}\right)^T & 0 \end{bmatrix}. \tag{16}$$

If $E^{(k)}$ is nonsingular when the approximate values $\lambda_j^{(0)}$ and $z_j^{(0)}$ in $E^{(k)}$ become the exact ones, each coefficient matrix encountred during the iteration process may be nonsingular. The resulting matrix $E$ will be

$$E = \begin{bmatrix} A - \lambda_j B & -B z_j \\ -\left(B z_j\right)^T & 0 \end{bmatrix}. \tag{17}$$

Consider the following new eigenproblem to prove the nonsingularity of $E$,

$$E\,\mu_i = \gamma_i\, F\,\mu_i \qquad i = 1, 2, \ldots, 2n, 2n+1, \tag{18}$$

where $\gamma_i$ and $\mu_i$ are the $i$ th eigenvalue and the associated eigenvector of the new eigenproblem, respectively, and

$$F=\begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} C & M & 0 \\ M & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{19}$$

or collectively

$$E\,U = F\,U\,\Gamma, \tag{20}$$

where

$$\Gamma = diag\begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{2n+1} \end{pmatrix} \quad \text{and} \quad U=\begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_{2n+1} \end{bmatrix}. \tag{22}$$

The eigenvalues and eigenvectors of the new eigenproblem are as follows;

- Eigenvalues :

$$\gamma's: \qquad -1, \qquad 1 \qquad \lambda_k - \lambda_j \; k=1,2,\ldots,2n, k\neq j\,. \tag{23}$$

- Eigenvectors :

$$\mu's: \quad \begin{Bmatrix} z_j \\ -1 \end{Bmatrix}, \qquad \begin{Bmatrix} z_j \\ 1 \end{Bmatrix} \qquad \begin{Bmatrix} z_k \\ 0 \end{Bmatrix} \; k=1,2,\ldots,2n, k\neq j\,. \tag{24}$$

Considering the determinant of eqn (20),

$$\det[E]\det[U]=\det[F]\det[U]\det[\Gamma], \tag{25}$$

or

$$\begin{aligned} \det[E] &= \det[F]\det[\Gamma] \\ &= -\det[F]\prod_{\substack{k=1 \\ k\neq j}}^{2n}(\lambda_k-\lambda_j), \end{aligned} \tag{26}$$

where

$$\begin{aligned} \det[F] &= \begin{vmatrix} C & M & 0 \\ M & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} C & M \\ M & 0 \end{vmatrix} \\ &= \det[-I]\det[M]\det[M]. \end{aligned} \tag{27}$$

The determinant of $F$ is not equal to zero because the determinant of $M$ is non-zero by definition. Supposing that all eigenvalues are distinct, the determinant of $E$ is not equal to zero since the determinant of $F$ is non-zero by eqn (27). The proof of the numerical stability is completed.

## NUMERICAL EXAMPLES

In this section, two test problems with nonclassically damping are presented to show the effectiveness of the proposed method. The symmetric Lanczos method[5] with partial reorthogonalization scheme is used to obtain the initial values of the proposed method. The lowest $p$ initial eigenpairs can be obtained by solving the standard eigenproblem reduced by

$2p$ Lanczos vectors. The approximate eigenpairs, then, are improved by the iterations until their error norm are within $10^{-6}$. The error norm is defined as

$$error\ norm = \frac{\left\| \left( A - \lambda_j^{(k)}\ B \right) z_j^{(k)} \right\|_2}{\left\| A\ z_j^{(k)} \right\|_2} \tag{28}$$

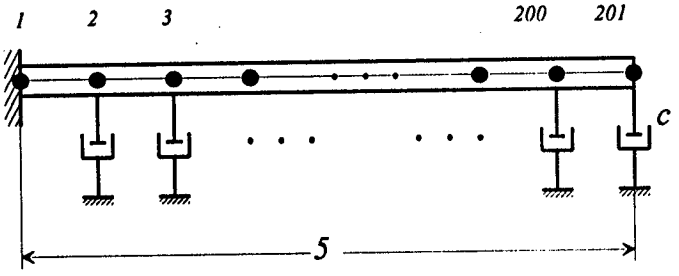
All executions are done on the CONVEX C3420 with 100 MIPS and 200 MFLOPS.

**Cantilever Beam with Multi-Lumped Dampers**

The example studied is a cantilever beam with multi-lumped translational viscous-dampers attached at each node. The geometrical configuration is shown in Fig. 1. The structural model could be considered as a representative of a soil-structure interaction. The structure is modeled by 100 equal elements and has 200 degrees of freedom. The order of the associated $(A, B)$ is 400. Dimensionless value of Young's modulus for the beam material is taken as 1000, while mass density, section area and inertia are specified to be of unit value. The mass matrix is consistent one. The damping matrix $C$ consists of the Rayleigh damping and concentrated dampers. The Rayleigh damping is a linear combination of the stiffness and mass matrices such as

$$C = \alpha\ M + \beta\ K, \tag{29}$$

where the Rayleigh coefficients $\alpha$ and $\beta$ are 0.001, respectively. On the other hand, the damping coefficients of the concentrated dampers are 0.1.




*TANGENTIAL DAMPER c : 0.1*
*YOUNG'S MODULUS : 1000*
*MASS DENSITY : 1*
*CROSS-SECTION INERTIA : 1*
*CROSS-SECTION AREA : 1*

Fig. 1 *Cantilever Beam with Multi-Lumped Dampers*

The results for the lowest five eigenpairs with error norm $10^{-6}$ are summarized in Table 2. The initial values of the proposed method which are calculated by the Lanczos method are about $10^{-1}$ to $10^{-3}$. Only one iteration is needed to obtain the eigenpairs with error norm $10^{-6}$. The CPU time for the proposed method is compared with that of the subspace iteration method in Table 3. If we let the solution time for the proposed method be 1, it takes 1.25 times for the subspace iteration method. The variations of the error norms for the each eigenpair corresponding to the iteration number are depicted in Figs. 2 to 6. The first step of the proposed method denotes the result of the Lanczos algorithm. The results indicate that the

convergence of the proposed method is much better than that of the subspace iteration method.

Table 2 *The results of the proposed method(Number of iterations, eigenvalue and error norm)*

| Mode number | Error norm of starting value (Lanczos method) | Proposed method | | |
|---|---|---|---|---|
| | | Number of iterations | Eigenvalue | Error norm |
| 1 | 0.872989E-04 | 1 | -1.02232 ± i 3.95028 | 0.183316E-07 |
| 2 | 0.763146E-03 | 1 | -1.18011 ± i 18.3991 | 0.189217E-09 |
| 3 | 0.437867E-04 | 1 | -1.79640 ± i 39.6535 | 0.373318E-10 |
| 4 | 0.605684E-02 | 1 | -2.87171 ± i 60.9945 | 0.371279E-11 |
| 5 | 0.420530E-00 | 1 | -4.40255 ± i 82.2930 | 0.983166E-07 |

Table 3 *CPU time spent for first five eigenpairs*

| Method | CPU time(in seconds) | Ratio |
|---|---|---|
| Subspace iteration method | 96.10 | 1.25 |
| Proposed method | 76.75 | 1.00 |
| ( Lanczos method + Iteration scheme ) | ( 10.55 + 66.20) | |

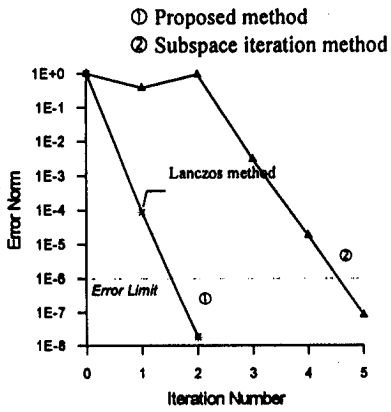



Fig.2 *Error norm of the 1st Eigenpair*

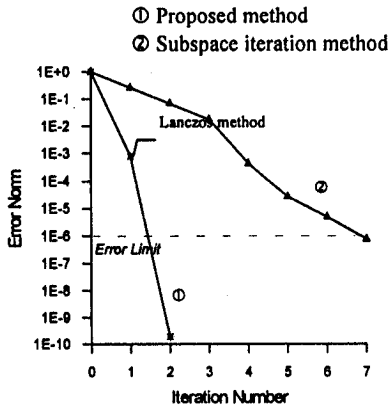



Fig.3 *Error norm of the 3rd Eigenpair*

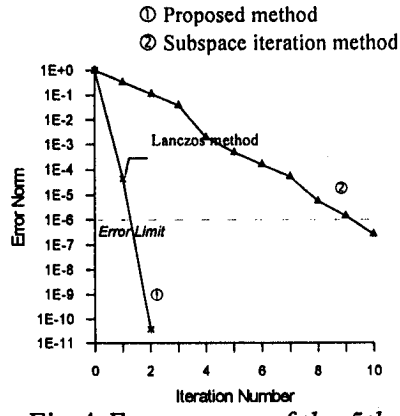



Fig.4 *Error norm of the 5th Eigenpair*

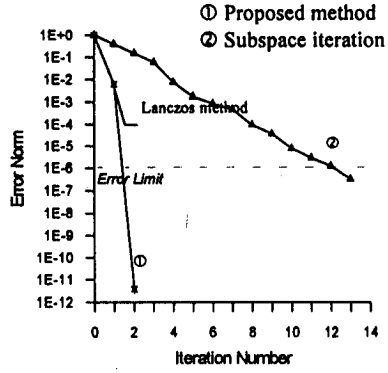



Fig.5 *Error norm of the 7th Eigenpair*

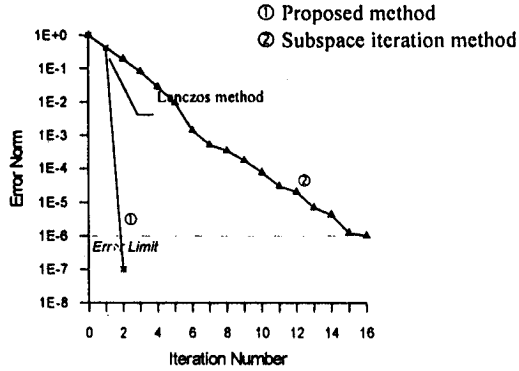
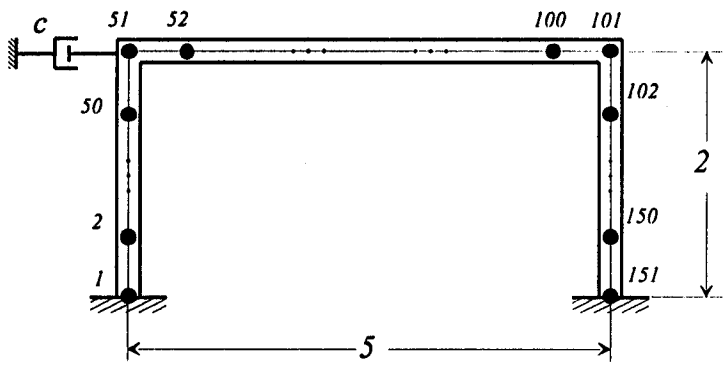



Fig.6 *Error norm of the 9th Eigenpair*

**Frame Structure with Concentrated Damper**

In this example, a frame structure with a concentrated damper is presented. The system could be considered as a representative of a control system or a passively damped space structure. The model is divided into 150 equal elements and has 447 degrees of freedom as shown in Fig. 7. The associated $(A, B)$ is of order 894. Dimensionless values for Young's modulus, density, inertia and area of beam are, respectively, 500, 1, 1, and 1. The consistent mass matrix is used to define $M$. The damping matrix $C$ consists of the Rayleigh damping and concentrated dampers. The Rayleigh coefficients $\alpha$ and $\beta$ are 0.001, respectively. The damping coefficient of the concentrated horizontal damper is 10.




*HORIZONTAL DAMPER c : 10*
*YOUNG'S MODULUS : 500*
*MASS DENSITY : 1*
*CROSS-SECTION INERTIA : 1*
*CROSS-SECTION AREA : 1*

Fig. 7 *Frame Structure with Concentrated Damper*

The results for the lowest five eigenpairs with error norm $10^{-6}$ are summarized in Table 4. The initial values of the proposed method which are calculated by the Lanczos method are about $10^{-0}$ to $10^{-6}$. The required number of iterations to satisfy the predetermined error norm is only one or two iterations despite of starting values with large error norms. For example the fifth starting values with error norm of $10^{-0}$ are converged to the values with the error norm of about $10^{-10}$ with only two iterations. The CPU time for the proposed method is compared with that of the subspace iteration method in Table 5. If we let the solution time for the proposed method be 1, it takes 1.18 times for the subspace iteration method. The variations of the error norms for the each eigenpair corresponding to the iteration number are depicted in Figs. 8 to 12. The first step of the proposed method denotes the result of the Lanczos algorithm. The results indicate that the convergence of the proposed method is much better than that of the subspace iteration method.

Table 4 *The results of the proposed method(Number of iterations, eigenvalue and error norm)*

| Mode number | Error norm of starting value (Lanczos method) | Proposed method | | |
|---|---|---|---|---|
| | | Number of iterations | Eigenvalue | Error norm |
| 1 | 0.169894E-05 | 1 | -0.06543 ± i 7.44209 | 0.483552E-10 |
| 2 | 0.274663E-04 | 1 | -0.39695 ± i 8.40284 | 0.165798E-09 |
| 3 | 0.193503E-01 | 1 | -0.07532 ± i 11.7071 | 0.200729E-10 |
| 4 | 0.732792E-01 | 1 | -0.71155 ± i 13.9090 | 0.447537E-10 |
| 5 | 1.000000E-00 | 2 | -0.46457 ± i 20.0691 | 0.300293E-10 |

Table 5 *CPU time spent for first five eigenpairs*

| Method | CPU time(in seconds) | Ratio |
|---|---|---|
| Subspace iteration method | 204.74 | 1.18 |
| Proposed method | 173.51 | 1.00 |
| ( Lanczos method + Iteration scheme ) | (14.24 + 159.27) | |

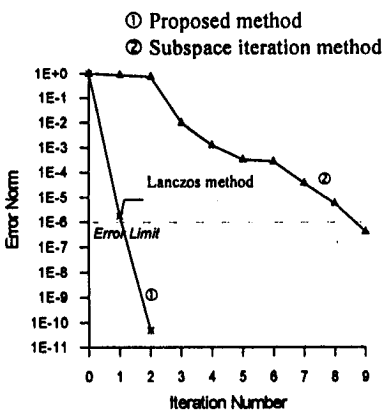



Fig.8 *Error norm of the 1st Eigenpair*

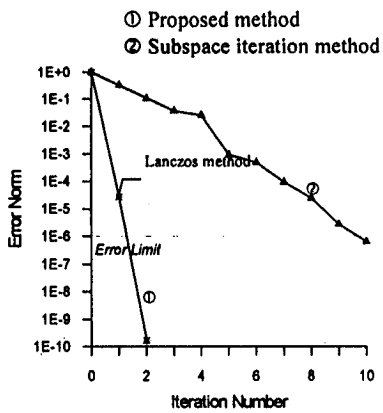



Fig.9 *Error norm of the 3rd Eigenpair*

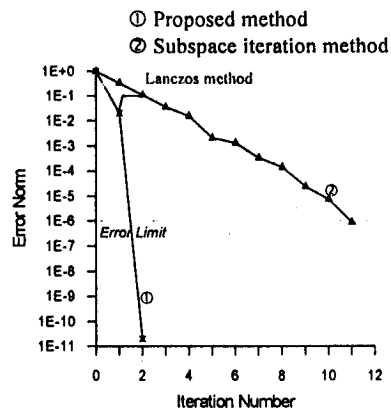



Fig.10 *Error norm of the 5th Eigenpair*

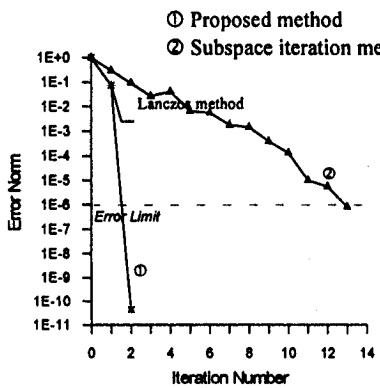



Fig.11 *Error norm of the 7th Eigenpair*

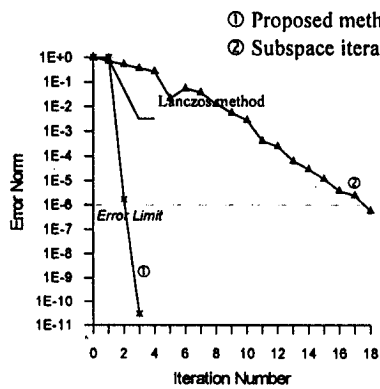



Fig.12 *Error norm of the 9th Eigenpair*

## CONCLUSIONS

In this paper, The method for finding the eigenpairs of the nonclassically damped systems is proposed. As shown in numerical analysis, section 3, characteristics of the proposed method are identified as follows;

① The proposed method has high convergence rate.

② The proposed method guarantees the numerical stability, which is proved analytically.

③ The proposed method takes full advantages of the sparseness of the system matrices.

## ACKNOWLEDGEMENT

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## REFERENCES

1. Caughey, T. K. and O'Kelly, M. E. J., "Classical normal modes in damped linear dynamic systems", Trans. ASME, J. appl. Mech., Vol. 32, 1965, pp. 583-588.
2. Chen, H. C., "Partial eigensolution of damped structural systems by Arnoldi's method", Earthqu. Engng Struct. Dyn.. Vol. 22, 1993, pp. 63-74.
3. Chen, H. C., "Efficient vibration analysis of general structural systems", Comput. Struct., Vol. 53, No. 5, 1994, pp. 1109-1114.
4. Chen, H. C. and Taylor, R. L., Properties and solutions of the eigensystem of non-proportionally damped linear dynamic systems, *Report No. UCB/SEMM-86/10, University of California at Berkeley*, 1986
5. Chen, H. C. and Taylor, R. L., "Solution of eigenproblems for damped structural systems by Lanczos algorithm", Comput. Struct., Vol. 30, No. 1/2, 1988, pp. 151-161.
6. Gupta, K. K., "Eigenproblem solution of damped structural systems", Int. J. Numer. Meth. Engng., Vol. 8, 1974, pp. 877-911.
7. Gupta, K. K., "Development of a unified numerical procedure for free vibration analysis of structures", Int. J. Numer. Meth. Engng., Vol. 17, 1981, pp. 187-198.
8. Kaufman, L., "LZ algorithm to solve the generalized eigenvalue problem", SIAM J. Numer. Anal., Vol. 11, 1974, pp. 997-1024.
9. Kim, H. M. and Craig, R. R. Jr, "Structural dynamics analysis using an unsymmetric block Lanczos algorithm", Int. J. Numer. Meth. Engng., Vol. 26, 1988, pp. 2305-2318.
10. Leung, A. Y. T., "Subspace iteration for complex symmetric eigenproblems", *J. sound vib.*, Vol. 184(2), 1995, pp. 627-637.
11. Moler, C. B. and Stewart, G. W., "An algorithm for generalized matrix eigenvalue problems", SIAM J. Numer. Anal., Vol. 10, 1973, pp. 241-256.
12. Parlett B. N. and Scott, D. S., "The Lanczos algorithm with selective orthogonalization", Math. Comput., Vol. 33, 1979, pp. 217-238.
13. Parlett, B. N., Taylor, D. R. and Liu, Z. A., "A look-ahead Lanczos algorithm for unsymmetric matrices", Math. Comp., Vol. 44, 1985, pp. 105-124.
14. Simon, H. D., "The Lanczos algorithm with partial reorthogonalization", Math. Comput., Vol. 42, 1984, pp. 115-142.
15. Veselic, L., "A global Jocobi method for a symmetric indefinite problem $S\,x = \lambda\, T\,x$", Comput. Meth. appl. Mech. Engng., Vol. 38, 1983, pp. 128-143.