An Efficient Algebraic Method for Computing Natural Frequency and Mode Shape Sensitivities of Eigenproblem with Distinct and Multiple Eigenvalues

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ABSTRACT

An efficient numerical method for computation of eigenpair derivatives for the real symmetric eigenvalue problem with distinct and multiple eigenvalues is presented. Using the proposed method, the exact eigenpair derivatives can be obtained by solving algebraic equations with symmetric coefficient matrix. As examples to demonstrate the efficiency of the proposed method a cantilever plate and a cantilever beam is considered. The results are compared with those of Nelson's method and Dailey's method, respectively.

INTRODUCTION

The behavior of many physical systems can be completely determined by obtaining the eigensolutions of the system. Variations in system parameters lead to changes in these eigensolutions. Knowledge of the eigenvector derivatives with respect to physical parameters can help an engineer optimize a structural design or minimize its sensitivity to the parameters. In addition, one can remarkably reduce the cost of reanalyzes using these eigenpair derivatives in large systems. Eigenpair derivatives are also useful in design trend studies and for gaining

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insight into the behavior of physical systems.

In 1976, Nelson presented an algorithm for computing the eigenvalue and eigenvector derivatives of general real matrices with non-repeated eigenvalues. Nelson's method, however, suffers from singularity problems when multiple eigenvalues exist. In typical structures, there are many multiple or nearly equal eigenvalues, due to structural symmetry. In 1989, Dailey presented an efficient method which amended Ojalvo's method(1988) for computing eigenvalue and eigenvector derivatives of real symmetric matrices with multiple eigenvalues. Other methods – modal method(Fox and Kapoor 1968), Wang's method(1985), Lim *et al.*(1987) and Liu *et al.*(1994) etc. – need all or must of the eigenvectors to find the eigenvector derivative.

The proposed method can find the eigenvector derivatives by solving the algebraic equations with a symmetric coefficient matrix which is added a side condition. The algorithm of the proposed method is very simple and compact. Furthermore, it can be saved the computer memory and analysis time. Note that the proposed method finds the exact eigenpair derivatives. Numerical examples are presented.

CURRENT METHODS FOR SENSITIVITY ANALYSIS

Theoretical Background

The real symmetric eigenvalue problem associated with linear vibration is defined as

$$K\phi_j = \lambda_j M\phi_j \ , \qquad (1)$$

where K and M are the stiffness and mass matrices, respectively; they are order n symmetric matrices. M is positive definite and K is positive definite or semi-positive definite. ϕ_j is the jth mode shape, λ_j is the square of the jth natural frequency. To obtain the derivative of the eigenvalue, Eq. (1) is differentiated with respect to a design parameter p,

$$(K-\lambda_j M)\phi'_j = -(K'-\lambda_j M')\phi_j + \lambda'_j M\phi_j . \qquad (2)$$

Premultiplying at each side of Eq. (2) by ϕ_j^T gives the eigenvalue derivative as

$$\lambda'_j = \phi_j^T (K'-\lambda_j M')\phi_j . \qquad (3)$$

The right side of Eq. (2) is known but the eigenvector derivative ϕ'_j cannot be found directly since the matrix $[\, K-\lambda_j M \,]$ is singular.

Nelson's Method : Case of Distinct Eigenvalues

In this method one expresses the eigenvector derivative in terms of a particular solution v_j and a homogeneous solution $c_j\phi_j$, where c_j is a coefficient to be determined.

$$\phi'_j = v_j + c_j\phi_j . \qquad (4)$$

The complete procedure of Nelson's method is summarized in Table 2.

Dailey's Method : Case of Multiple Eigenvalues

When the eigenvalues are repeated ones and a design parameter is perturbed, the eigenvectors split into as many as m distinct eigenvectors. For the derivatives of the eigenvectors to exist, the eigenvectors must be adjacent to the m distinct eigenvectors that appear when the design parameter varies. Consider the following eigenvalue problem and an orthogonal transformation, in which Ψ, of order $(n \times m)$, is a matrix of eigenvectors with multiple eigenvalues, and Γ is an orthonormal transformation matrix.

$$K\Psi = M\Psi\Lambda \quad \text{and} \quad Z = \Psi\Gamma . \tag{5, 6}$$

Here $\Lambda = \lambda I_m$ and $\Psi^T M \Psi = I_m$, where λ is the eigenvalue of multiplicity m, and Z is adjacent eigenvector set. The derivatives of the adjacent eigenvectors can be expressed as.

$$Z' = V + ZC .$$

The procedure for finding the particular solution set and the homogeneous solution set is summarized in Table 3 and 4, respectively.

PROPOSED METHOD FOR CALCULATING EIGENPAIR DERIVATIVES

Case of Distinct Eigenvalues

The proposed method finds the eigenvector derivative by solving the linear algebraic equations with symmetric coefficient matrix. Rewriting Eq. (2),

$$(K - \lambda_j M)\phi'_j - \lambda'_j M\phi_j = -(K' - \lambda_j M')\phi_j . \tag{7}$$

Differentiating the mass normalized equation with respect to the design parameter,

$$\phi_j^T M \phi'_j + 0.5\phi_j^T M' \phi_j = 0 . \tag{8}$$

Eq. (7) and Eq. (8) may be written as a single matrix equation as follows:

$$\begin{bmatrix} K - \lambda_j M & -M\phi_j \\ -\phi_j^T M & 0 \end{bmatrix} \begin{Bmatrix} \phi'_j \\ \lambda'_j \end{Bmatrix} = \begin{Bmatrix} -(K' - \lambda_j M')\phi_j \\ 0.5\phi_j^T M' \phi_j \end{Bmatrix} . \tag{9}$$

The derivatives ϕ'_j and λ'_j can be found by solving Eq. (9). The algorithm of the proposed method is very simple and compact. The numerical stability of the proposed method can be shown very easily.

Case of Multiple Eigenvalues

The adjacent eigenvectors must be calculated first to apply the proposed method in the case of multiple eigenvalues. Considering the following eigenvalue problem with the normalization condition to obtain the eigenvector derivatives:

$$KZ = MZ\Lambda \quad \text{and} \quad Z^T M Z = I_m . \tag{10, 11}$$

Differentiating Eq. (10) and Eq. (11) with respect to the design parameter p, and writing in a single matrix equation:

$$\begin{bmatrix} K - \lambda M & -MZ \\ -Z^T M & 0 \end{bmatrix} \begin{bmatrix} Z' \\ \Lambda' \end{bmatrix} = \begin{bmatrix} -(K' - \lambda M')Z \\ 0.5Z^T M' Z \end{bmatrix} . \tag{12}$$

The derivatives Z' and Λ' can be found by solving Eq. (12).

NUMERICAL EXAMPLES

Cantilever Plate : Case of Distinct Eigenvalues

As a simple application, the cantilever plate shown in Figure 1 is considered. The total degrees of freedom of the FE model is 936. Young's modulus is $10.5\times10^5\ N/m^2$, the mass density $5.88\times10^{-3}\ kg/m^2$ and Poisson's ratio 0.3. The length of the plate is 6 *m*, and width 3 *m* and thickness 0.01 *m*. The design parameter is the plate thickness *t*. Table 1 shows the first ten eigenvalues and their derivatives with respect to the thickness of the cantilever plate. The analysis time of the proposed method is compared with that of Nelson's method. The total central processor seconds of the proposed method is 46.8 sec, and Nelson's method 47.3 sec. The CP time of the proposed method takes similar to that of Nelson's method since the proposed method requires the solution of $(n+1)$ simultaneous equations, whereas Nelson's method computes the constant *c* and the linear symmetric algebraic equation of order $(n-1)$. All runs are executed in IRIS4D-20-S17 which has 10 Mips and 0.9 MFlops.

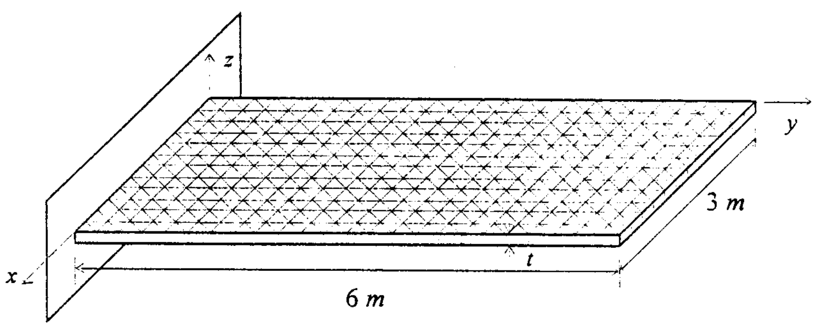

Figure 1. Cantilever plate with the thickness *t* as the design parameter

Cantilever Beam : Case of Multiple Eigenvalues

Consider the cantilever beam with square section shown in Figure 2. The FE model has 800 degrees of freedom. Young's modulus is $2.10\times10^{11}\ N/m^2$, mass density $7.85\times10^3\ kg/m^3$ and Poisson's ratio 0.3. Both the beam height and width are 0.1 *m*, and its length 10 *m*. The design parameter is the beam height *h*.

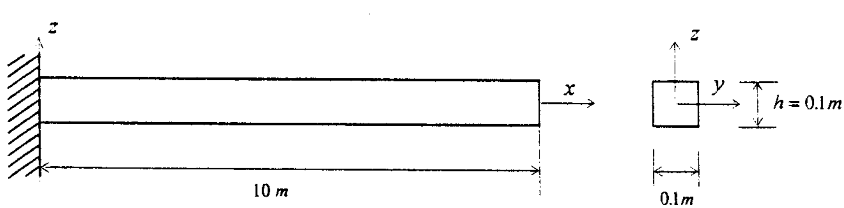

Figure 2. Cantilever beam with the height *h* as the design parameter

Table 1 shows the first ten eigenvalues and their derivatives with respect to the height of the cantilever beam. The analysis time of the proposed method is compared with that of Dailey's method. The total CP time of the proposed method is 7.2 sec, and Dailey's method 10.1 sec. It can be reduced the analysis time about 30 %. Furthermore, it can be saved the computer space about 33 %.

Table 1. *The first ten eigenvalues and their derivatives with respect to the thickness and the height of the cantilever plate and the cantilever beam, respectively.*

Mode number	Cantilever Plate		Cantilever Beam	
	Eigenvalues	Eigenvalue Derivatives	Eigenvalues	Eigenvalue Derivatives
1	0.33872085 E+02	0.67744170 E+04	0.68831827 E+03	0.36816052 E−03
2	0.62019287 E+03	0.12403658 E+06	0.68831827 E+03	0.27532760 E+04
3	0.13098816 E+04	0.26197632 E+06	0.26877592 E+05	0.35624980 E−04
4	0.65670945 E+04	0.13134189 E+07	0.26877592 E+05	0.10751037 E+06
5	0.10268439 E+05	0.20536878 E+07	0.20878576 E+06	0.25249360 E−03
6	0.23767562 E+05	0.47535123 E+07	0.20878576 E+06	0.83514303 E+06
7	0.24200821 E+05	0.48401644 E+07	0.79110317 E+06	0.10139465 E−02
8	0.39428178 E+05	0.78856356 E+07	0.79110317 E+06	0.31644127 E+07
9	0.44496013 E+05	0.88992027 E+07	0.21251636 E+07	0.60539320 E−03
10	0.65517630 E+05	0.13103526 E+08	0.21251636 E+07	0.85006543 E+07

CONCLUSIONS

This paper proposes an efficient numerical method for calculating the eigenpair derivatives with respect to the design parameter by solving the linear algebraic equation without any numerical instability. The proposed method has the desirable properties of preserving the structure of the stiffness and mass matrices, and of requiring knowledge of only eigenpairs to be differentiated. The proposed method does not need to find the second order derivative of the stiffness and mass matrices in the eigenvalue problem with the multiple eigenvalues. The analysis time is saved about 30 % because the complicated procedure for determining the coefficient matrix C and the clumsy procedure for finding the particular solution V encountered in Dailey's method do not needed in the proposed method.

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Table 2. *The procedure of Nelson's method*

(1) Compute $\lambda'_j = \phi_j^T (K' - \lambda_j M')\phi_j$.
(2) Let $f_j = -(K' - \lambda_j M')\phi_j + \lambda'_j M\phi_j$, and $G_j \equiv K - \lambda_j M$.
(3) Find k such that ϕ_{jk} is the largest element on the column of ϕ_j.
(4) Construct $\overline{G}_j$ by zeroing out row k and column k of G_j and setting the kth diagonal to 1.
(5) Construct $\bar{f}_j$ by zeroing out the kth element of f_j.
(6) Solve $\overline{G}_j v_j = \bar{f}_j$.
(7) Compute $c_j = -v_j^T M\phi_j - 0.5\phi_j^T M'\phi_j$.
(8) Let $\phi'_j = v_j + c_j\phi_j$.

Table 3. *The procedure for finding the particular solution V.*

(1) Let $Z = [z_{ij}]$
(2) Find k such that $z_{1,k}$ is the largest element on the first column of Z.
(3) Replace the kth row and column of G with zeros, and setting the kth diagonal to 1.
(4) Replace the kth row of F with zeros.
(5) Go back to step 2 and repeat for the next column of Z until through.
(6) Call the resulting matrices $\overline{G}$ and $\overline{F}$.
(7) Solve $\overline{G}V = \overline{F}$.

Table 4. *The algorithm of Dailey's method.*

(1) Compute $G = K - \lambda M$ and $F \equiv (\lambda M' - K')Z + MZ\Lambda'$.
(2) Find the m rows of Z containing the largest elements. Zero out these rows and columns of G and the same rows of F. Place 1 in the affected diagonal elements of G and call the resulting matrices $\overline{G}$ and $\overline{F}$.
(3) Solve $\overline{G}V = \overline{F}$.
(4) Compute $Q = C + C^T = -V^T MZ - Z^T MV - Z^T M'Z$.
(5) Compute $R = C\Lambda' - \Lambda'C + 0.5\Lambda'' = Z^T(K' - \lambda M')V - Z^T(M'Z + MV)\Lambda' + 0.5Z^T(K'' - \lambda M'')Z$.
(6) Construct the $m \times m$ matrix C by the rule

$$c_{ij} = \begin{Bmatrix} r_{ij}/(\lambda'_j - \lambda'_i), & \text{if } \lambda'_j \neq \lambda'_i \\ 0.5\, q_{ij}, & \text{otherwise} \end{Bmatrix}, \text{ where } \Lambda' = diag(\lambda'_1, \cdots, \lambda'_m).$$

(7) Let $Z' = V + ZC$. The columns of Z' are the eigenvector derivatives.