

## An Efficient Solution Method of Generalized Eigenproblems with Multiple Eigenvalues

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### ABSTRACT

An efficient solution method is presented to solve the eigenvalue problem arising in the dynamic analysis of nonclassically damped structural systems with multiple eigenvalues. The proposed method is obtained by applying the modified Newton-Raphson technique and the orthonormal condition of the eigenvectors to the quadratic eigenvalue problem. In the iteration methods such as the inverse iteration method and the subspace iteration method, singularity may be occurred during the factorizing process when the shift value is close to an eigenvalue of the system. However, even if the shift value is an eigenvalue of the system, the proposed method guarantees nonsingularity, which is analytically proved. A numerical example is also presented to demonstrate the effectiveness of the proposed method and the results are compared with those of the well-known subspace iteration method and the Lanczos method.

### INTRODUCTION

In the analysis of dynamic response of structural systems, the equation of motion of damped systems can be written as

$$M \ddot{u}(t) + C \dot{u}(t) + K u(t) = f(t), \quad (1)$$

where  $M$ ,  $K$  and  $C$  are the ( $n$  by  $n$ ) mass, stiffness and nonproportional damping matrices,

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respectively, and  $\ddot{u}(t)$ ,  $\dot{u}(t)$  and  $u(t)$  are the ( $n$  by 1) acceleration, velocity and displacement vectors, respectively. To find the solution of free vibration of the system, we consider the following quadratic eigenproblem

$$\lambda^2 M \phi + \lambda C \phi + K \phi = 0 \quad (2)$$

in which  $\lambda$  and  $\phi$  are the eigenvalue and eigenvector of the system. There are  $2n$  eigenvalues for the system with  $n$  degrees of freedom and these occur either in real pairs or in complex conjugate pairs, depending upon whether they correspond to overdamped or underdamped modes.

The common practice is to reformulate the quadratic system of equation to a linear one by doubling the order of the system such as

$$A \psi = \lambda B \psi \quad (3)$$

where

$$A = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \quad B = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \quad \text{and} \quad \psi = \begin{Bmatrix} \phi \\ \lambda \phi \end{Bmatrix} \quad (4)$$

The classical inverse iteration method (Gupta 1974) is commonly used to solve for only a small number of desired modes. The subspace iteration method (Leung 1995) is a more efficient alternative than the inverse iteration method. However, in the iteration methods such as the inverse iteration method and the subspace iteration method, a large number of complex arithmetic are required. Furthermore, when the shift value becomes close to an eigenvalue of the system, singularity may be encountered during triangularization process.

The Lanczos algorithm for the computation of eigenvalues and eigenvectors of a real symmetric matrix was proposed by Lanczos in 1950 and improved by numerous researchers (Paige 1972, etc.). The Lanczos algorithm to solve the eigenvalue problem of nonclassically damped system is developed by Rajakumar and Rogers (two sided-Lanczos algorithm) and by Chen and Taylor (symmetric Lanczos algorithm). Although only real arithmetic is solved during the solution process, in contrast to the case of real symmetric eigenproblems, there will be a possibility of serious breakdown and the accuracy of the solutions obtained is low (Zheng, Ren and Wang 1997).

In this paper, the method to solve an eigenproblem with guaranteed nonsingularity for a damped structural system with multiple eigenvalues is developed.

## METHOD OF ANALYSIS

### Problem Definition

We consider an eigenproblem of which the eigenvalue  $\lambda_i$  has multiplicity  $m$ . For simplicity let us assume that the first  $m$  eigenvalues are equal

$$\lambda \equiv \lambda_1 = \lambda_2 = \dots = \lambda_m. \quad (5)$$

Then Eq. (3) can be presented in matrix form for the  $m$  multiple eigenvalues as follows

$$A \Psi = B \Psi \Lambda \quad (6)$$

where  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m) = \lambda I_m$  and  $\Psi = [\psi_1 \ \dots \ \psi_m]$  is a ( $n$  by  $m$ ) matrix satisfying the orthonormal condition with respect to matrix  $B$  such as

$$\Psi^T B \Psi = I_m \quad (7)$$

where  $I_m$  is an unitary matrix of order  $m$ .

The objective is to develop an efficient solution method with guaranteed non-singularity for an eigenproblem described by Eq. (6) and Eq. (7).

### Proposed Method

Let us assume that initial approximate solutions of Eq. (6)  $\Lambda^{(0)}$  and  $\Psi^{(0)}$  are known. Denoting the approximate eigenvalues and the associated eigenvectors after  $k$  iterations by  $\Lambda^{(k)}$  and  $\Psi^{(k)}$ , we can get

$$R^{(k)} = A \Psi^{(k)} - B \Psi^{(k)} \Lambda^{(k)} \quad (8)$$

and

$$(\Psi^{(k)})^T B \Psi^{(k)} = I_m \quad (9)$$

where the residual matrix  $R^{(k)} = [r_1^{(k)} \ \dots \ r_m^{(k)}]$  denotes the error for each eigenpair, and is not generally zero because of substitution of approximate values into Eq. (6).

In order to get the solutions converged to the multiple eigenvalues and the associated eigenvectors of the system, the residual vectors should be removed. For the purpose of that, the Newton-Raphson technique is applied such as

$$\begin{aligned} R^{(k+1)} &= A \Psi^{(k+1)} - B \Psi^{(k+1)} \Lambda^{(k+1)} \\ &= 0 \end{aligned} \quad (10)$$

and

$$(\Psi^{(k+1)})^T B \Psi^{(k+1)} = I_m \quad (11)$$

where

$$\Lambda^{(k+1)} = \Lambda^{(k)} + \Delta\Lambda^{(k)} \quad (12)$$

$$\Psi^{(k+1)} = \Psi^{(k)} + \Delta\Psi^{(k)} \quad (13)$$

Substituting Eq. (12) and Eq. (13) into Eq. (10) and Eq. (11) and neglecting the nonlinear terms  $B \Delta\Psi^{(k)} \Delta\Lambda^{(k)}$  and  $(\Delta\Psi^{(k)})^T B \Delta\Psi^{(k)}$ , we can get the linear simultaneous equations for unknown incremental values  $\Delta\Lambda^{(k)}$  and  $\Delta\Psi^{(k)}$  as follows;

$$A \Delta\Psi^{(k)} - B \Delta\Psi^{(k)} \Lambda^{(k)} - B \Psi^{(k)} \Delta\Lambda^{(k)} = -R^{(k)} \quad (14)$$

and

$$(\Psi^{(k)})^T B \Delta\Psi^{(k)} = 0. \quad (15)$$

Since the eigenvalue is multiple, the off-diagonal elements of  $\Lambda^{(k)}$  are zero or very small compared with its diagonal at  $k$ th iteration step, and the diagonal element very close. Thus, the second term in left side of Eq. (14) may be approximated by  $\lambda_1^{(k)} B \Delta\Psi^{(k)}$ , which yields

$$A \Delta \Psi^{(k)} - \lambda_1^{(k)} B \Delta \Psi^{(k)} - B \Psi^{(k)} \Delta \Lambda^{(k)} = -R^{(k)} \quad (16)$$

Writing Eq. (16) and Eq. (15) in matrix form, we can get

$$\begin{bmatrix} (A - \lambda_1^{(k)} B) & -B\Psi^{(k)} \\ (-B\Psi^{(k)})^T & 0 \end{bmatrix} \begin{bmatrix} \Delta\Psi^{(k)} \\ \Delta\Lambda^{(k)} \end{bmatrix} = - \begin{bmatrix} R^{(k)} \\ 0 \end{bmatrix} \quad (17)$$

Because the new coefficient matrix should be reformed and refactorized in each iteration step, the above method adopting the Newton-Raphson technique and a side condition, despite of its rapid convergence, is not efficient.

These blemishes may be overcome by applying the modified Newton-Raphson technique to Eq. (17) such as

$$\begin{bmatrix} (A - \lambda_1^{(0)} B) & -B\Psi^{(k)} \\ (-B\Psi^{(k)})^T & 0 \end{bmatrix} \begin{bmatrix} \Delta\Psi^{(k)} \\ \Delta\Lambda^{(k)} \end{bmatrix} = - \begin{bmatrix} R^{(k)} \\ 0 \end{bmatrix} \quad (18)$$

The symmetric coefficient matrix of Eq. (18) is of order  $(2n + m)$ . While singularity occurs in factorization process of the iteration methods such as the inverse iteration method and the subspace iteration method when the shift is close to an eigenvalue of the system, nonsingularity of the proposed method is always guaranteed (Lee, Kim and Robinson 1997). This is the main difference compared with the iteration method with shift.

Initial values of the proposed method can be obtained as the intermediate results of the iteration methods or results of approximate methods. In this paper, the starting values are taken as the results of the symmetric Lanczos method (Chen and Taylor 1988) with selectively reorthogonalization process because the method does not need complex arithmetic in the Lanczos recursive process, and because the multiplicity of the desired eigenvalues can be checked by the results of the  $4p$  Lanczos vectors ( $p$ : the number of desired eigenvalues).

## NUMERICAL EXAMPLE

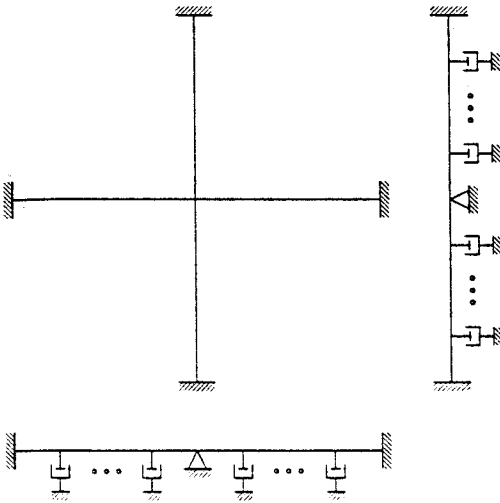
In this section a test problem with multiple eigenvalues is used to assess the performance of the proposed method for generalized eigenproblems. The CPU time spent for the first twelve eigenvalues and the associated eigenvectors ( $p=12$ ) and the variation of the error norm to each iteration step of the proposed method are compared with those of the subspace iteration method (Leung 1995). The least subspace dimension to effectively calculate required eigenpairs is  $2p$  (24 eigenpairs). Each method is stopped when the error norms are reduced by the factor of  $10^{-6}$ , which yields a stable eigensolution and sufficient accuracy in the calculated eigenvalues and eigenvectors for practical analysis (Bathe and Ramaswamy 1980). The error norm (Bathe and Ramaswamy 1980) is defined as

$$\varepsilon_i^{(k)} = \frac{\|r_i^{(k)}\|_2}{\|A \psi_i^{(k)}\|_2} \quad (19)$$

where

$$R^{(k)} = [r_1^{(k)} \quad \dots \quad r_m^{(k)}] = A\Psi^{(k)} - B\Psi^{(k)}\Lambda^{(k)} \quad (20)$$

## Plane Frame Structure with Lumped Dampers



Young's Modulus : 1000    Mass Density : 1.0  
 Cross-section Inertia : 1.0    Cross-section Area : 1.0  
 Proportional Damping Coeff.:  $\alpha = 0.001$ ,  $\beta = 0.001$   
 Concentrated Damping : 0.3

Fig.1 Plane frame structure with lumped dampers

indicate that the convergence of the proposed method is much better than that of the subspace iteration method. The results of the Lanczos method in Fig. 4 are not improved in spite of the increase of the number of the Lanczos vectors. The CPU time for the proposed method is compared with that of the subspace iteration method in Table 1. If we let the solution time for the proposed method be 1, it takes 3.55 times for the subspace iteration method.

## CONCLUSIONS

An efficient method for solving damped structural dynamic eigenproblems with multiple eigenvalues is presented. Characteristics of the proposed method identified by the numerical results from a test problem are identified as follows;

- ① Since the convergence rate of the proposed method is high, the proposed method is very effective for solving damped dynamic systems with a large number of degrees of freedom.
- ② Nonsingularity of the proposed method is always guaranteed, which is proved analytically.
- ③ The algorithm of the proposed method is very simple.

## REFERENCES

- Bathe, K.J. and Ramaswamy, S. (1980), "An accelerated subspace iteration method", *Comput. Struct.*, **23**, 313-331.
- Chen, H.C. and Taylor, R.L. (1988), "Solution of eigenproblems for damped structural systems by Lanczos algorithm", *Comput. Struct.*, **30**(1/2), 151-161.

The finite element model of a plane frame is used as the first example. The geometric configuration and material properties are shown in Fig. 1. The model is discretized in 200 beam elements resulting in the system of dynamic equations with a total of 590 degrees of freedom. Thus, the order of the associated eigenproblem is 1180. The consistent mass matrix is used for  $M$ . Its damping matrix is derived from the proportional damping expression given by  $C = \alpha K + \beta M$  and concentrated dampers.

All the eigenvalues of the model are multiple. The variations of the error norms to increasing the iteration step are shown in Figs 2 to 4. The error norms of the initial values obtained by using the  $4p(48)$  Lanczos vectors are about  $0.7$  to  $10^{-7}$ , which are possible to check the multiplicity of the desired eigenvalues. The number of iterations for the proposed method applied to the initial values which does not satisfy the error norm  $10^{-6}$  is only one. The results in Figs 2 to 3

Gupta, K.K. (1974), "Eigenproblem solution of damped structural systems", *Int. J. Numer. Meth. Engng.*, **8**, 877-911.

Leung, A.Y.T. (1995), "Subspace iteration method for complex symmetric eigen-problems", *J. Sound Vibr.*, **184**(4), 627-637.

Lee, I.W., Kim, M.C. and Robinson, A.R. (1997), "An efficient solution method of eigenproblems for damped structural systems using the modified Newton-Raphson technique", *J. Engng. Mech.*, ASCE, accepted.

Paige, C.C. (1972), "Computational variants of the Lanczos method for the eigenproblem", *J. Inst. Math. Appl.*, **10**, 373-381.

Rajakumar, C. and Rogers, C.R. (1991), "The Lanczos algorithm applied to unsymmetric generalized eigenvalue problem", *Int. J. Numer. Meth. Engng.*, **32**, 1009-1026.

Zheng, Z.C., Ren, G.X. and Wang, W.J. (1997), "A reduction method for large scale unsymmetric eigenvalue problems in structural dynamics", *J. Sound Vibr.*, **199**(2), 253-268.

Table 1. CPU time spent for the first twelve eigenvalues of the plane frame structure with lumped dampers

Methods	CPU time(in seconds), Ratio
Proposed method	872.69 (1.00)
Subspace iteration method	3096.62 (3.55)

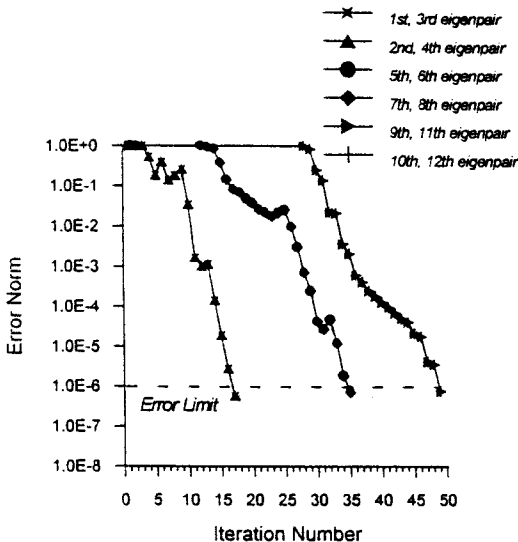


Fig. 3 Variation of the error norm of the beam model by the subspace iteration method

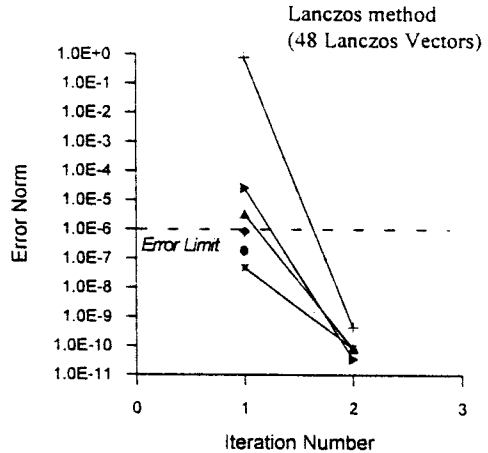


Fig. 2 Variation of the error norm of the beam model by the proposed method

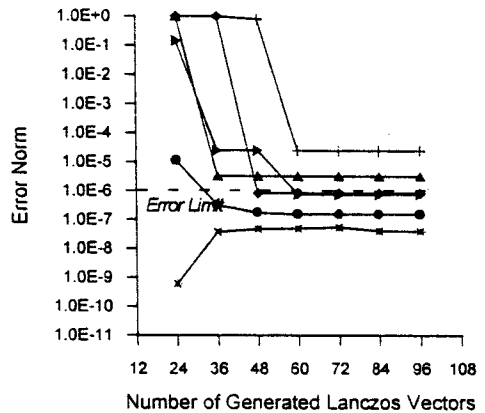


Fig. 4 Variation of the error norm of the beam model by the Lanczos method