

Efficient Eigensolution Method of Large Structural Systems with Proportional and Non-Proportional Dampers

H.-J. Jung and I.-W. Lee, M. ASCE

Korea Advanced Institute of Science and Technology, Taejeon 305-701, Korea
s_hjjung@kaist.ac.kr, iwlee@kaist.ac.kr

Abstract:

An efficient numerical method for computing the eigenproblem for the large structural systems with proportional or non-proportional dampers is presented. The proposed method is obtained by applying the modified Newton-Raphson technique and the orthonormal condition of the eigenvectors. In the iteration methods such as the inverse iteration method and the subspace iteration method, singularity may be occurred during the factorizing process when the shift value is very close to an eigenvalue of the system. However, even if the shift value is an eigenvalue of the system itself, the proposed method guarantees nonsingularity, which is analytically proved. The initial values of the proposed method can be taken as the intermediate results of iteration methods or results of approximate methods. The efficiency of the proposed method is verified by comparing convergence and solution time for numerical examples with those of the subspace iteration method, the determinant search method and the Lanczos method.

Introduction

The dynamic equation of motion can be written as in equation (1)

$$M \ddot{u}(t) + C \dot{u}(t) + K u(t) = f(t) \quad (1)$$

where M , C and K are the mass, damping and stiffness matrices of order n , respectively, $u(t)$ the displacement vector, and $f(t)$ the load vector.

The methods for solving dynamic equation of motion can be divided into the step by step integration method and the mode superposition method. If the dynamic analysis is performed by the mode superposition method, the free vibration analysis must be first solved. And most of computational time is required for free vibration analysis. Therefore, an efficient eigensolution technique is necessarily required.

For symmetric and positive definite matrices, the eigenvalues and the associated eigenvectors of the system in equation (1) may be determined in a straightforward and efficient manner provided that the damping matrix is proportional or, more generally (Caughey and O'Kelly (1965)), provided that

$$CM^{-1}K = KM^{-1}C. \quad (2)$$

That is, in case of structures with proportional dampers, we can get easily the desired eigenpairs by analyzing the following eigenproblem of order n not considering the damping matrix C ;

$$K \phi_i = \lambda_i M \phi_i \quad (3)$$

where λ_i is the i th eigenvalue and ϕ_i the corresponding eigenvector.

Systems for which equation (2) is not satisfied are called the non-proportionally

damped. The eigenanalysis for such systems is traditionally performed in the space extended to $2n$ -dimensional such as

$$\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \phi_i \\ \lambda_i \phi_i \end{Bmatrix} = \lambda_i \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} \phi_i \\ \lambda_i \phi_i \end{Bmatrix}. \quad (4)$$

Method of Analysis

We consider an eigenproblem of which the eigenvalue λ_i has multiplicity m . For simplicity let us assume that the first m eigenvalues are equal or close to each other

$$\lambda \equiv \lambda_1 \cong \lambda_2 \cong \dots \cong \lambda_m. \quad (5)$$

If the multiplicity of an eigenproblem equals 1, then the eigenvalues of the eigenproblem are not multiple or close, but distinct.

The eigenvalue problem of proportionally and non-proportionally damped system can be presented in matrix form for the m multiple or close eigenvalues as follows;

$$A \Psi = B \Psi \Lambda. \quad (6)$$

For proportionally damped case, $(n \times n)$ matrices A , B and $(n \times 1)$ vectors ψ_i ($i = 1, \dots, m$) can be expressed as follows;

$$A = K, \quad B = M \quad \text{and} \quad \psi_i = \phi_i. \quad (7)$$

For non-proportionally damped case, $(2n \times 2n)$ matrices A , B and $(2n \times 1)$ vectors ψ_i ($i = 1, \dots, m$) can be written like this;

$$A = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \quad B = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \quad \text{and} \quad \psi_i = \begin{Bmatrix} \phi_i \\ \lambda_i \phi_i \end{Bmatrix}, \quad (8)$$

where $\Psi = [\psi_1 \ \dots \ \psi_m]$ is a matrix satisfying the orthonormal condition with respect to matrix B such as

$$\Psi^T B \Psi = I_m \quad (9)$$

and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m) = \lambda I_m$ is a matrix of order m .

Let $X = [x_1, \dots, x_m]$ be the vectors in the subspace Ψ , and X be the orthonormal with respect to B . Then

$$\Psi = XZ \quad (10)$$

$$X^T B X = I_m \quad (11)$$

where Z is the unknown rotation matrix of order m .

Introducing equation (10) into equation (6), we get

$$AXZ = BXZ\Lambda. \quad (12)$$

Let

$$DZ = Z\Lambda \quad (13)$$

where $D = [d_1, d_2, \dots, d_m] = X^T A X$ and symmetric.

Then,

$$AXZ = BXDZ \quad (14)$$

and

$$AX = BXD \quad (15)$$

or

$$Ax_i = BXd_i \quad (i = 1, \dots, m) . \quad (16)$$

We obtain the m multiple or close eigenvalues and associated eigenvectors from equations (10), (13) and (15).

Note when m eigenvalues are multiple ($\lambda_1 = \dots = \lambda_m$), from equation (13)

$$\begin{aligned} \Lambda &= D \\ \Psi &= X , \end{aligned}$$

and when m eigenvalues are close to each other ($\lambda_1 \approx \dots \approx \lambda_m$), D is not a diagonal matrix. We should solve the small standard eigenvalue problem as in equation (13). Then we get the eigenvalues Λ from equation (13) and the eigenvectors Ψ from equation (10).

Let us assume that initial approximate solutions of equation (6), $\Lambda^{(0)}$ and $\Psi^{(0)}$, are known. Denoting the approximate eigenvalues and the associated eigenvectors after k iterations by $\Lambda^{(k)}$ and $\Psi^{(k)}$, we can get

$$R^{(k)} = A \Psi^{(k)} - B \Psi^{(k)} \Lambda^{(k)} \quad (17)$$

and

$$(\Psi^{(k)})^T B \Psi^{(k)} = I_m \quad (18)$$

where the residual matrix $R^{(k)} = [r_1^{(k)} \quad \dots \quad r_m^{(k)}]$ denotes the error for each eigenpair, and is not generally zero because of substitution of approximate values into equation (6).

In order to get the solutions converged to the multiple eigenvalues and the associated eigenvectors of the system, the residual vectors should be removed. For the purpose of that, the Newton-Raphson technique is applied such as

$$\begin{aligned} R^{(k+1)} &= A \Psi^{(k+1)} - B \Psi^{(k+1)} \Lambda^{(k+1)} \\ &= 0 \end{aligned} \quad (19)$$

and

$$(\Psi^{(k+1)})^T B \Psi^{(k+1)} = I_m \quad (20)$$

where

$$\Lambda^{(k+1)} = \Lambda^{(k)} + \Delta\Lambda^{(k)} \quad (21)$$

$$\Psi^{(k+1)} = \Psi^{(k)} + \Delta\Psi^{(k)} . \quad (22)$$

Substituting equation (21) and equation (22) into equation (19) and equation (20) and neglecting the nonlinear terms, $B \Delta\Psi^{(k)} \Delta\Lambda^{(k)}$ and $(\Delta\Psi^{(k)})^T B \Delta\Psi^{(k)}$, we can get the linear simultaneous equations for unknown incremental values, $\Delta\Lambda^{(k)}$ and $\Delta\Psi^{(k)}$ as follows;

$$A \Delta\Psi^{(k)} - B \Delta\Psi^{(k)} \Lambda^{(k)} - B \Psi^{(k)} \Delta\Lambda^{(k)} = -R^{(k)} \quad (23)$$

and

$$(\Psi^{(k)})^T B \Delta\Psi^{(k)} = 0 . \quad (24)$$

Since the eigenvalue is multiple, the off-diagonal elements of $\Lambda^{(k)}$ are zero or very

small compared with its diagonal at k th iteration step, and the diagonal element very close. Thus, the second term in left side of equation (23) may be approximated by $\lambda_1^{(k)} B \Delta\Psi^{(k)}$, which yields

$$A \Delta\Psi^{(k)} - \lambda_1^{(k)} B \Delta\Psi^{(k)} - B \Psi^{(k)} \Delta\Lambda^{(k)} = -R^{(k)}. \quad (25)$$

Writing equation (24) and equation (25) in matrix form, we can get

$$\begin{bmatrix} (A - \lambda_1^{(k)} B) & -B\Psi^{(k)} \\ (-B\Psi^{(k)})^T & 0 \end{bmatrix} \begin{bmatrix} \Delta\Psi^{(k)} \\ \Delta\Lambda^{(k)} \end{bmatrix} = - \begin{bmatrix} R^{(k)} \\ 0 \end{bmatrix}. \quad (26)$$

Because the new coefficient matrix should be reformed and refactorized in each iteration step, the above method adopting the Newton-Raphson technique and a side condition, despite of its rapid convergence, is not efficient.

These blemishes may be overcome by applying the modified Newton-Raphson technique to equation (26) such as

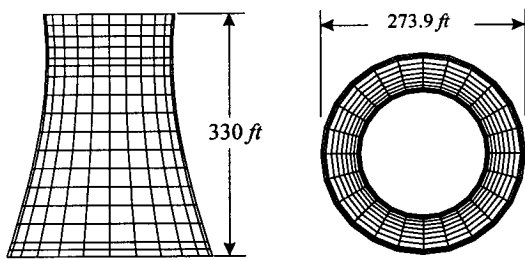
$$\begin{bmatrix} (A - \lambda_1^{(0)} B) & -B\Psi^{(k)} \\ (-B\Psi^{(k)})^T & 0 \end{bmatrix} \begin{bmatrix} \Delta\Psi^{(k)} \\ \Delta\Lambda^{(k)} \end{bmatrix} = - \begin{bmatrix} R^{(k)} \\ 0 \end{bmatrix}. \quad (27)$$

The symmetric coefficient matrix of equation (27) is of order $(n+m)$ in case of proportional damping system, and of order $(2n+m)$ in case of non-proportional damping system. While singularity occurs in factorization process of the iteration methods such as the inverse iteration method and the subspace iteration method when the shift is close to an eigenvalue of the system, nonsingularity of the proposed method is always guaranteed (Lee, Kim and Robinson (1998)). This is the main difference compared with the iteration methods with shift.

Numerical Examples

Proportionally damped case: Cooling tower structure

The cooling tower structure shown Figure 1 consists of 408 four-node shell elements, 432 nodes and 2,448 degrees-of-freedom. The stiffness and the mass matrix have the mean half-bandwidths of 201. The first twenty eigenvalues of the model are double roots.



Elevation

Plan

Young's modulus: 4.32×10^8 psf

Mass density: 4.66 slugs/ft³

Poisson ratio: 0.15

Shell thickness: 0.583 ft

Figure 1. Cooling tower structure

Each solution time for two solution methods to get 10 and 20 eigenpairs with the error norm of 10^{-6} and 10^{-9} is summarized in Table 1, in which we check the solution time and the convergence rate especially in the case of 10 eigenpairs with error norm 10^{-9} . Determinant search method is not applied because it did not give us the good results. If we let the solution time for the proposed method be 1, it takes 2.0 times for

the accelerated subspace iteration method. For each solution method the convergence of eigenpairs is represented from Figure 2 to Figure 3.

Table 1. CPU time (in seconds) of the cooling tower structure

Analysis Methods	$p = 10$		$p = 20$	
	$EN=10^{-6}$	$EN=10^{-9}$	$EN=10^{-6}$	$EN=10^{-9}$
Proposed Method	2785.8 (1.0)	3067.7 (1.0)	5104.2 (1.0)	5576.5 (1.0)
Subspace Iteration Method	4584.9 (1.6)	6182.5 (2.0)	6383.6 (1.3)	15829.3(2.8)
Determinant Search Method	No Sol.	No Sol.	No Sol.	No Sol.

$p = \text{no. of the desired eigenpairs, } EN = \text{Error Norm}$

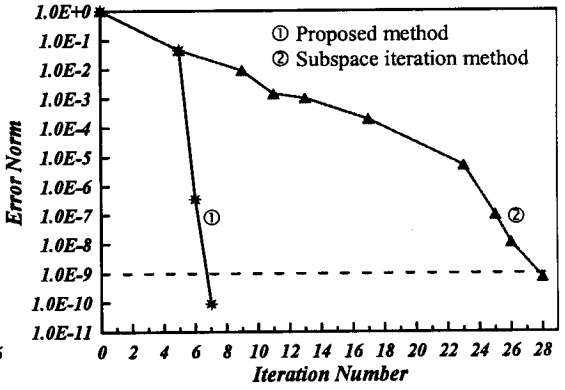
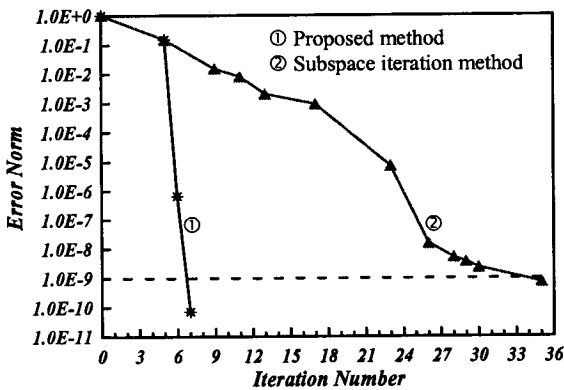


Figure 2. Convergence of the 8th eigenpair

Figure 3. Convergence of the 10th eigenpair

Non-proportionally damped case: Plane frame structure with lumped dampers

The model is discretized in 200 beam elements resulting in the system of dynamic equations with a total of 590 degrees of freedom. Thus, the order of the associated eigenproblem is 1180. The consistent mass matrix is used for M . Its damping matrix is derived from the proportional damping expression given by $C = \alpha M + \beta K$ and concentrated dampers. the first twelve eigenvalues ($p=12$) of the model are double roots.

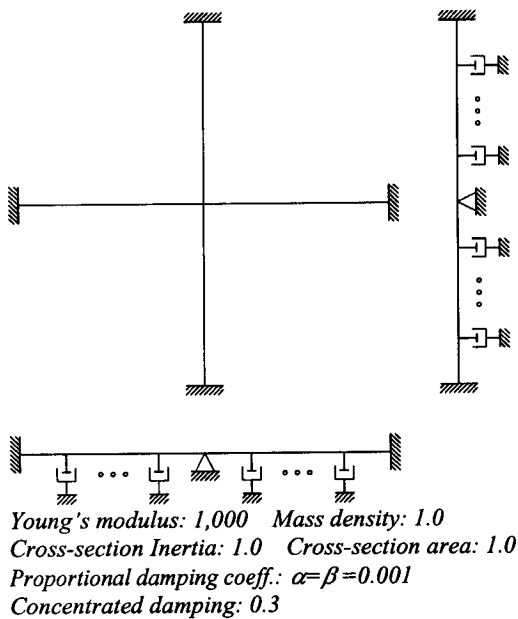


Figure 4. Plane frame structure with lumped dampers

The CPU time spent for the first twelve eigenvalues of the proposed method is compared with that of the subspace iteration method in Table 2. If we let the solution time for the proposed method be 1, it takes 3.55 times for the subspace iteration method. The variations of the error norms to increasing the iteration step are shown in Figures 5 to 7. The error

norms of the initial values obtained by using the $4p$ (48) Lanczos vectors are about 0.7 to 10^{-7} , which are possible to check the multiplicity of the desired eigenvalues. The number of iterations for the proposed method applied to the initial values that do not satisfy the error norm 10^{-6} is only one. The results in Figures 5 to 6 indicate that the convergence of the proposed method is much better than that of the subspace iteration method. The results of the Lanczos method in Figure 7 are not improved in spite of the increase of the number of the Lanczos vectors.

Table 2. CPU time (in seconds) of the plane frame structure with lumped dampers

Methods	CPU time (Ratio)
Proposed method	872.69 (1.00)
Subspace iteration method	3096.62 (3.55)

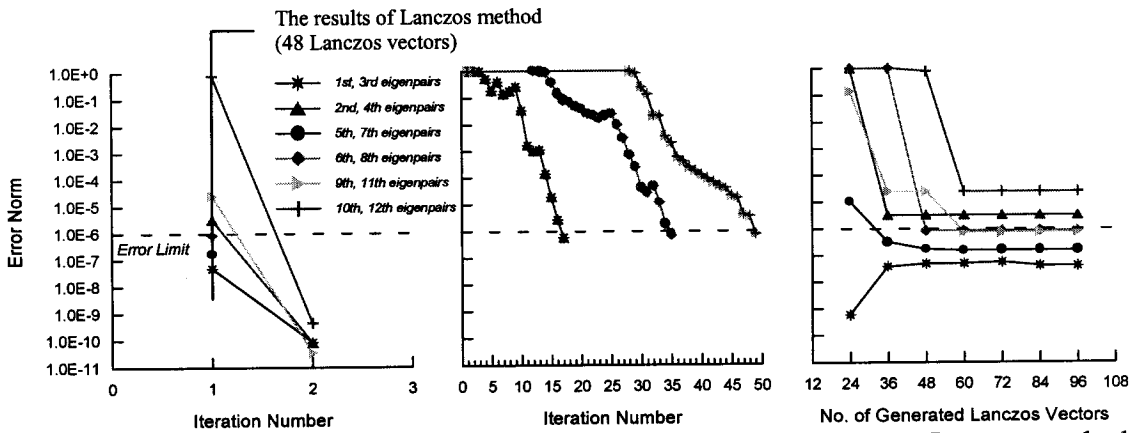


Figure 5. Proposed method

Figure 6. Subspace iteration method

Figure 7. Lanczos method

Conclusions

An efficient method for solving eigenproblems of structures with proportional and non-proportional dampers is presented. Characteristics of the proposed method identified by the numerical results from a test problem are identified as follows;

- ① Since the proposed method converges very fast, the proposed method is very effective for solving dynamic systems with a large number of degrees of freedom.
- ② Nonsingularity of the proposed method is always guaranteed, which is proved analytically.

References

- Caughey, T.K. and O’Kelly, M.E.J. (1965). “Classical normal modes in damped linear dynamic systems.” *Trans. ASME, J. Appl. Mech.*, **32**, 583-588.
- Lee, I.-W., Kim, M.-C., and Robinson, A.R. (1998). “An efficient solution method of eigenproblems for damped structural systems using the modified Newton-Raphson technique.” *J. Engng. Mech.*, ASCE, **124**(5), 576-580.