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ANALYSIS AND COMPARISON OF STEP-BY-STEP NUMERICAL INTEGRATION METHODS

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ABSTRACT: Step-by-step numerical integration methods are analyzed and compared. The amplification matrix is constructed according to algorithm of each method. Stability and accuracy analysis is done by finding eigenvalues of the amplification matrix and compared with one another. Four story shear building under external excitation is analyzed by each method and the displacements of top floor are compared with exact solutions.

KEYWORDS: step-by-step numerical integration method, time increment, amplification matrix, stability and accuracy analysis

1. INTRODUCTION

It is common to use step-by-step numerical integration methods for dynamic analysis of structures. The well-known methods of them are central difference method, Houbolt method [1], Newmark method [2] and Wilson θ method [3]. And many other methods have been developed and modified until now [4,5,6,7,8]. In order to choose the effective step-by-step numerical integration method, comparison of all the methods is required, but has not been performed yet. In this paper, most of step-by-step numerical integration methods are compared.

Since the cost of step-by-step numerical integration methods is directly proportional to the number of time steps, time increment should be large enough to save cost, whereas time increment should be small enough to obtain accurate solution. So we should select appropriate time increment compromising two contradictory criteria. Stability and accuracy analysis is helpful to select appropriate time increment. Stability and accuracy analysis is performed by eigenvalues of amplification matrix obtained by solving free vibration analysis of SDOF system using algorithm of each method.

Numerical example of four story shear building under sinusoidal loading and El Centro earthquake is performed by each method and the results are compared with exact solutions.

2. AMPLIFICATION MATRIX

Step-by-step numerical integration methods transfer the state at the n th step to the (n+1) th step and this can be written as follows:

$$\{X_{n+1}\} = [A]\{X_n\} + [L]\{f_{n+v}\} \tag{1}$$

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where $\{X_n\}$ is the displacement, velocity or acceleration solved last step . $[A]$ is the amplification matrix which transfer $\{X_n\}$ to the next step. $\{f_{n+v}\}$ is external force at each step and $[L]$ is load factor to connect external force and $\{X_{n+1}\}$. The amplification matrix of each method is obtained by solving the equation of motion of SDOF system in discrete time domain, which can be written as follows:

$$\ddot{x}_n + 2\xi\omega\dot{x}_n + \omega^2 x_n = f_n \tag{2}$$

The amplification matrix of some methods is shown as follows.

- Central Difference Method

$$\begin{Bmatrix} x_{n+1} \\ x_n \end{Bmatrix} = \begin{bmatrix} \dfrac{2-\omega_n^2\Delta t^2}{1+\xi\omega_n\Delta t} & -\dfrac{1-\xi\omega_n\Delta t}{1+\xi\omega_n\Delta t} \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} x_n \\ x_{n-1} \end{Bmatrix}$$

- Houbolt Method

$$\begin{Bmatrix} x_{n+1} \\ x_n \\ x_{n-1} \end{Bmatrix} = \begin{bmatrix} \dfrac{5\lambda}{\omega_n^2\Delta t^2}+6\kappa & -\left(\dfrac{4\lambda}{\omega_n^2\Delta t^2}+3\kappa\right) & \dfrac{\lambda}{\omega_n^2\Delta t^2}+\dfrac{2\kappa}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \end{Bmatrix}$$

where

$$\lambda = \left(\frac{2}{\omega_n^2\Delta t^2} + \frac{11\xi}{3\omega_n\Delta t} + 1\right)^{-1}, \kappa = \frac{\xi\lambda}{\omega_n\Delta t}$$

- Newmark Method

$$\begin{Bmatrix} x_{n+1} \\ \dot{x}_{n+1} \\ \ddot{x}_{n+1} \end{Bmatrix} = \begin{bmatrix} 1-\beta\lambda & \Delta t(1-\beta\lambda-2\beta\kappa) & \Delta t^2\left[\dfrac{1}{2}-\beta-\left(\dfrac{1}{2}-\beta\right)\beta\lambda-2(1-\gamma)\beta\kappa\right] \\ \dfrac{-\lambda\gamma}{\Delta t} & 1-\lambda\gamma-2\gamma\kappa & \Delta t\left[1-\gamma-\left(\dfrac{1}{2}-\beta\right)\gamma\lambda-2(1-\gamma)\gamma\kappa\right] \\ \dfrac{-\lambda}{\Delta t^2} & \dfrac{-(\lambda+2\kappa)}{\Delta t} & -\left(\dfrac{1}{2}-\beta\right)\lambda-2(1-\gamma)\kappa \end{bmatrix} \begin{Bmatrix} x_n \\ \dot{x}_n \\ \ddot{x}_n \end{Bmatrix}$$

where

$$\lambda = \left(\frac{1}{\omega_n^2\Delta t^2} + \frac{2\xi\gamma}{\omega_n\Delta t} + \beta\right)^{-1}, \kappa = \frac{\xi\lambda}{\omega_n\Delta t}$$

- Wilson θ Method

$$\begin{Bmatrix} x_{n+1} \\ \dot{x}_{n+1} \\ \ddot{x}_{n+1} \end{Bmatrix} = \begin{bmatrix} 1-\dfrac{\lambda}{6} & \Delta t\left(1-\dfrac{\lambda\theta}{6}-\dfrac{\kappa}{3}\right) & \Delta t^2\left(\dfrac{1}{2}-\dfrac{1}{6\theta}-\dfrac{\lambda\theta^2}{18}-\dfrac{\kappa\theta}{6}\right) \\ \dfrac{-\lambda}{2\Delta t} & 1-\dfrac{\lambda\theta}{2}-\kappa & \Delta t\left(1-\dfrac{1}{2\theta}-\dfrac{\lambda\theta^2}{6}-\dfrac{\kappa\theta}{2}\right) \\ \dfrac{-\lambda}{\Delta t^2} & \dfrac{-(\lambda\theta+2\kappa)}{\Delta t} & 1-\dfrac{\lambda\theta^2}{3}-\dfrac{1}{\theta}-\kappa\theta \end{bmatrix} \begin{Bmatrix} x_n \\ \dot{x}_n \\ \ddot{x}_n \end{Bmatrix}$$

where

$$\lambda = \left(\frac{\theta}{\omega_n^2\Delta t^2} + \frac{\xi\theta^2}{\omega_n\Delta t} + \frac{\theta^3}{6}\right)^{-1}, \kappa = \frac{\xi\lambda}{\omega_n\Delta t}$$

- Collocation Method

$$\begin{Bmatrix} x_{n+1} \\ \dot{x}_{n+1} \\ \ddot{x}_{n+1} \end{Bmatrix} = \begin{bmatrix} 1-\beta\lambda & \Delta t(1-\beta(\lambda\theta+2\kappa)) & \Delta t^2\left(\dfrac{1}{2}-\dfrac{\beta}{\theta}-\beta\theta^2\left(\dfrac{1}{2}-\beta\right)\lambda-2\beta(1-\gamma)\kappa\theta\right) \\ \dfrac{-\lambda}{2\Delta t} & 1-\dfrac{\lambda\theta}{2}-\kappa & \Delta t\left(1-\dfrac{\gamma}{\theta}-\theta^2\left(\dfrac{1}{2}-\beta\right)\gamma\lambda-2\gamma(1-\gamma)\kappa\theta\right) \\ \dfrac{-\lambda}{\Delta t^2} & \dfrac{-(\lambda\theta+2\kappa)}{\Delta t} & 1-\theta^2\left(\dfrac{1}{2}-\beta\right)\lambda-\dfrac{1}{\theta}-2(1-\gamma)\kappa\theta \end{bmatrix} \begin{Bmatrix} x_n \\ \dot{x}_n \\ \ddot{x}_n \end{Bmatrix}$$

where

$$\lambda=\left(\frac{\theta}{\omega_n^2\Delta t^2}+\frac{2\xi\gamma\theta^2}{\omega_n\Delta t}+\theta^3\beta\right)^{-1}\quad,\kappa=\frac{\xi\lambda}{\omega_n\Delta t}$$

- Hilber-Hughes-Taylor α Method

$$\begin{Bmatrix}x_{n+1}\\ \dot{x}_{n+1}\\ \ddot{x}_{n+1}\end{Bmatrix}=\begin{bmatrix}1-\beta\lambda & \Delta t(1-(1+\alpha)\beta\lambda-2\beta\kappa) & \Delta t^2\left[\frac{1}{2}-\beta-(1+\alpha)\left(\frac{1}{2}-\beta\right)\beta\lambda-2(1+\alpha)(1-\gamma)\beta\kappa\right]\\ \frac{-\lambda\gamma}{\Delta t} & 1-(1+\alpha)\lambda\gamma-2\gamma\kappa & \Delta t\left[1-\gamma-(1+\alpha)\left(\frac{1}{2}-\beta\right)\gamma\lambda-2(1+\alpha)(1-\gamma)\gamma\kappa\right]\\ \frac{-\lambda}{\Delta t^2} & \frac{-((1+\alpha)\lambda+2\kappa)}{\Delta t} & -(1+\alpha)\left(\frac{1}{2}-\beta\right)\lambda-2(1+\alpha)(1-\gamma)\kappa\end{bmatrix}\begin{Bmatrix}x_n\\ \dot{x}_n\\ \ddot{x}_n\end{Bmatrix}$$

where $\lambda=\left(\frac{1}{\omega_n^2\Delta t^2}+\frac{2(1+\alpha)\xi\gamma}{\omega_n\Delta t}+(1+\alpha)\beta\right)^{-1}\quad,\kappa=\frac{\xi\lambda}{\omega_n\Delta t}$

- Bozzak-Newmark Method

$$\begin{Bmatrix}x_{n+1}\\ \dot{x}_{n+1}\\ \ddot{x}_{n+1}\end{Bmatrix}=\begin{bmatrix}1-\beta\lambda & \Delta t(1-\beta\lambda-2\beta\kappa) & \Delta t^2\left[\frac{1}{2}-\beta\kappa-\frac{\beta\lambda}{2}-\frac{\beta\lambda}{\omega_n^2\Delta t^2}\right]\\ \frac{-\lambda\gamma}{\Delta t} & 1-\lambda\gamma-2\gamma\kappa & \Delta t\left[1-\gamma\kappa-\frac{\gamma\lambda}{2}-\frac{\gamma\lambda}{\omega_n^2\Delta t^2}\right]\\ \frac{-\lambda}{\Delta t^2} & \frac{-(\lambda+2\kappa)}{\Delta t} & 1+\kappa-\frac{\lambda}{2}-\frac{\lambda}{\omega_n^2\Delta t^2}\end{bmatrix}\begin{Bmatrix}x_n\\ \dot{x}_n\\ \ddot{x}_n\end{Bmatrix}$$

where

$$\lambda=\left(\frac{1-\alpha}{\omega_n^2\Delta t^2}+\frac{2\xi\gamma}{\omega_n\Delta t}+\beta\right)^{-1}\quad,\kappa=\frac{\xi\lambda}{\omega_n\Delta t}$$

- Zienkiewicz Method

$$\begin{Bmatrix}x_{n+1}\\ \dot{x}_{n+1}\end{Bmatrix}=\begin{bmatrix}1-\alpha\lambda & \Delta t(1-2\kappa\alpha-\lambda\alpha\theta)\\ -\frac{\lambda}{\Delta t} & 1-2\kappa-\lambda\theta\end{bmatrix}\begin{Bmatrix}x_n\\ \dot{x}_n\end{Bmatrix}$$

where $\lambda=\left(\frac{1}{\omega_n^2\Delta t^2}+\frac{2\xi\theta}{\omega_n\Delta t}+\alpha\theta\right)^{-1}\quad,\kappa=\frac{\xi\lambda}{\omega_n\Delta t}$

- Bazzi-Anderheggen ρ Method

$$\begin{Bmatrix}x_{n+1}\\ \dot{x}_{n+1}\\ \ddot{x}_{n+1}\end{Bmatrix}=\begin{bmatrix}1-\frac{\lambda}{2} & \lambda & \frac{\lambda(\rho-1)}{2(\rho+1)}\\ \frac{-\lambda}{\Delta t} & 1-\frac{\lambda}{1+\rho}-2\kappa & \Delta t\left(\frac{1-\rho}{1+\rho}\left(\kappa+\frac{\lambda}{2(1+\rho)}\right)\right)\\ \frac{-\lambda}{2\Delta t^2}(1+\rho)(1+\rho^2) & \frac{-(1+\rho^2)}{\Delta t}\left((1+\rho)\kappa+\frac{\lambda}{2}\right) & \frac{1-\rho-\rho^2-\rho^3}{2}+\frac{\kappa(1+\rho-\rho^2-\rho^3)}{2}+\frac{\rho^2(1-\rho)\lambda}{4(1+\rho)}\end{bmatrix}\begin{Bmatrix}x_n\\ \dot{x}_n\\ \ddot{x}_n\end{Bmatrix}$$

where, $\lambda=\left(\frac{1}{\omega_n^2\Delta t^2}+\frac{\xi}{\omega_n\Delta t}+\frac{1}{2(1+\rho)}\right)^{-1}\quad,\kappa=\frac{\xi\lambda}{\omega_n\Delta t}$

3. STABILITY AND ACUURACY ANALYSIS

Stability and accuracy analysis can be performed by solving eigenproblem of the amplification matrix. Stability criterion is that

$$\rho(\lambda)\le 1 \tag{3}$$

where λ is eigenvalue of amplification matrix and ρ(λ) is spectral radius whic' is maximum absolute value of all eigenvalues. Algorithmic damping ratio and relative period error are calculated by Equation (4) and Equation (5) in order to do accuracy analysis.

$$\bar{\xi}=-\frac{\ln(A^2+B^2)}{2\bar{\Omega}} \tag{4}$$

$$\tau=\frac{\omega}{\bar{\omega}}-1 \tag{5}$$

where $$\lambda_{1,2} = A \pm iB \, , \overline{\Omega} = \frac{\arctan(B/A)}{\sqrt{1-\xi^2}} \, , \overline{\omega} = \frac{\overline{\Omega}}{\Delta t} \quad (6)$$

Spectral radius, algorithmic damping ratio and relative period error of each method are shown as Figure 1,2 and 3. The x-axis is Δt/T where T is the smallest period of structure in these figures.

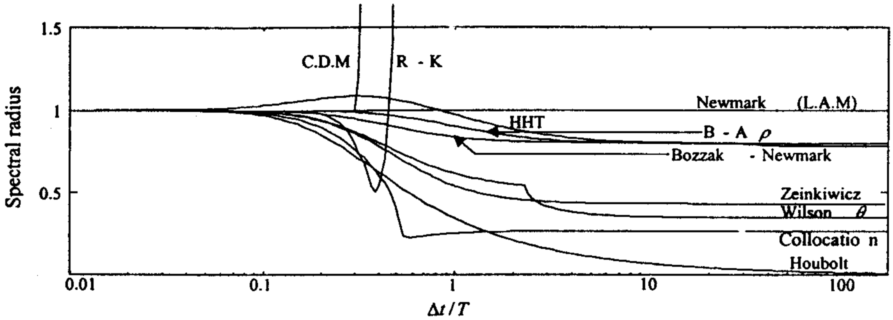

Figure 1. Comparison of spectral radii of each method with no damping

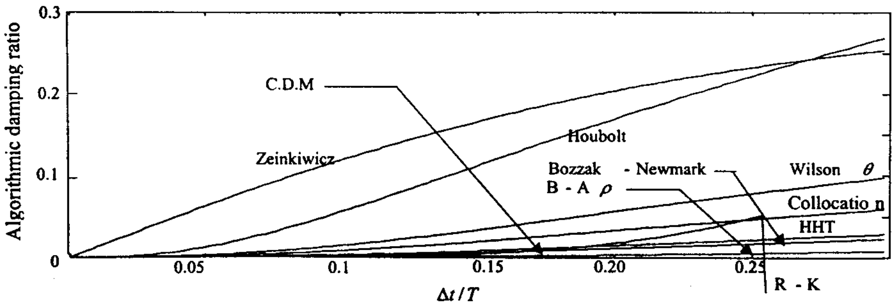

Figure 2. Comparison of algorithmic damping ratios of each method with no damping

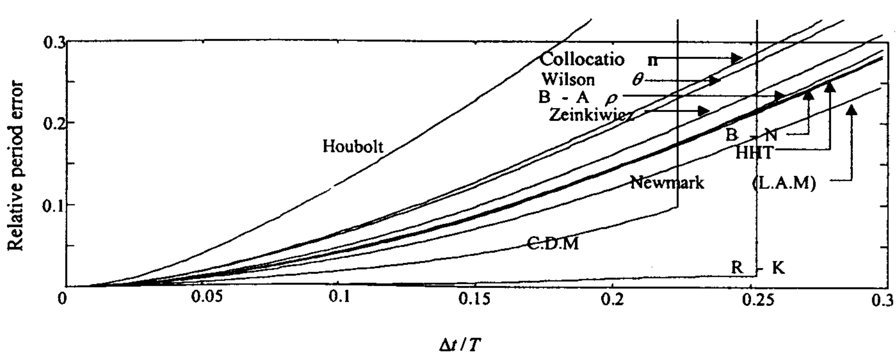

Figure 3. Comparison of relative period errors with no damping

4. NUMERICAL EXAMPLE : FOUR STORY SHEAR BUILDING

The four story shear building is shown as Figure 4. The damping matrix C consists of the Rayleigh damping. The Rayleigh damping coefficient α, β are 0.918 and 0.0023 respectively. The building is excited by sinusoidal loading with frequency of 7 rad/s, 20 rad/s and El Centro earthquake at foundation. By using of each step-by-step numerical integration method, the displacement of top floor is solved. The displacement solved analytically by mode superposition method using all modes is regarded as the exact solutions. The root-mean-square (RMS) values of Equation (7) are compared.

$$\text{RMS} = \frac{\sqrt{\sum (x_i - \bar{x}_i)^2}}{\text{No. of steps}} \tag{7}$$

where x_i is exact solution and $\bar{x}_i$ is calculated value by each method. The results of step-by-step numerical integration methods are shown as Figure 5.

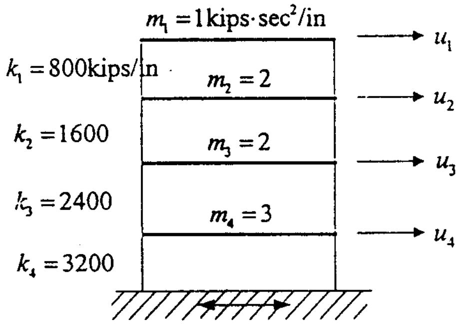

Figure 4. Four story shear building [11]

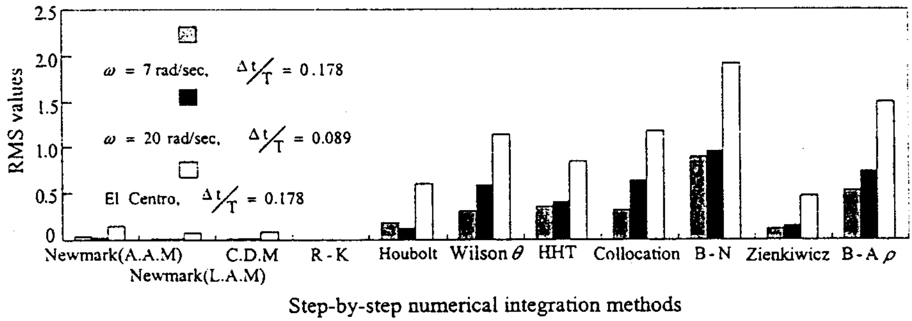

Figure 5. Comparison of RMS errors of step-by-step integration method

5. CONCLUSIONS

Step-by-step numerical integration methods are investigated. Stability and accuracy analysis of each method is performed analytically. The dynamic analysis of four story shear building is done numerically and compared with the exact solutions.

The conclusions of the step-by-step numerical integration methods identified by the analytic and numerical results can be summarized as follows:

1) Newmark method, central difference method and Runge-Kutta method have shown very precise results in Figure 5. The dynamic analysis of structure using these methods is performed with little error if Δt/T is less than 0.15.

2) Houbolt method and Zienkiwicz method have shown relatively accurate results. These methods may give good results in most case if Δt/T is less than 0.15.

3) In spite of good result of stability and accuracy analysis, Wilson θ method, Hilber-Hughes-Taylor α

method, collocation mathod, Bozzak-Newmak method and Bazzi-Anderheggen ρ method had some errors. Therefore, parameters of these methods should be selected carefully though Δt/T is enough small (less than 0.15).

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