Proceedings of the Twelfth KKNN Seminar/Workshop on Civil Engineering
KAIST, Taejon, Korea, August 20-22, 1999

EFFICIENT MODE SUPERPOSITION METHODS FOR NON-CLASSICALLY DAMPED SYSTEM

Sang-Won CHO[1], In-Won LEE[2] and Ju-Won OH[3]

ABSTRACT : The mode acceleration and the modal truncation augmentation methods are expanded to non-classically damped system for a modal response analysis of the structures with non-classically damping. The mode acceleration and the modal truncation augmentation methods improve the simple mode superposition method by complementing the portion of the truncated high modes but were used only in classically damped systems. To apply these methods to non-classically damped systems, the mode acceleration and the modal truncation augmentation methods are expanded to the non-classically damped systems in the state space approach. The applicability of expansion is verified by the closed form solutions and the numerical examples. The expanded modal truncation augmentation method is conditionally stable depending on the pattern of the external loading whereas the expanded mode acceleration method is stable for the all cases of loading in the non-classically damped systems. In the stable case, the results of the expanded modal truncation augmentation method are the same with that of the expanded mode acceleration method.

KEYWORDS : *Mode superposition method, Non-proportional damping, Mode acceleration method,* Modal truncation method

1. INTRODUCTION

Dynamic analysis of the structures is carried out in two ways. One way is the direct integration method and the other is the mode superposition method. Direct integration method is more accurate but needs high computation costs because full model of structure is used. The mode superposition method reduces model to a much smaller number of dynamic degrees of freedom prior to solving for the responses. Therefore the mode superposition method is more efficient than the direct integration method. Unfavorably, the reduction process of the mode superposition method alters the modal representation of loading and can adversely affect the quality of the calculated responses. The mode acceleration (MA) and the modal truncation augmentation (MT) methods improve the quality of the simple mode superposition method by complementing the portion of the truncated high modes.

In dynamic analysis of linear system, such as structure subjected to seismic excitations, it is common to assume that the system is classically damped. Under this assumption, the MA and MT methods are used. However, in most real systems, the modal equations are coupled by the damping matrix. In practice, non-classically damped system can be approximated by a classically damped system neglecting the coupling damping terms.[1] However, there are important situations where the effect of non-classically damping is essential and must be included in the analysis such as in soil-structure systems. A general method of analysis of multi-degree-of-freedom (MDOF) dynamic system is the state space approach.[2] This approach is applicable to systems with classically and non-classically damping not neglecting the coupling damping. In this paper for an efficient modal analysis, the mode acceleration and the modal truncation augmentation methods are expanded to the non-classically damped systems under the state space approach. The applicability of expansion is verified by the closed form solutions and the numerical examples.

[1] Dept. of Civil Engineering, KAIST, Korea, Graduate Student
[2] Dept. of Civil Engineering, KAIST, Korea, Professor
[3] Dept. of Civil & Environmental Engineering, Hannam University, Korea, Professor

2.MODE SUPERPOSITION METHOD FOR NON-CLASSICALLY DAMPED SYSTEM

Let the discrete equations of motion of a linear structural system be given by:

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{C}\dot{\mathbf{u}}(t)+\mathbf{K}\mathbf{u}(t)=\mathbf{R}_0 r(t) \quad (1)$$

where M, C and K are the n by n mass, damping and stiffness matrices, respectively; $\ddot{\mathbf{u}}(t)$, $\dot{\mathbf{u}}(t)$ and $\mathbf{u}(t)$ are the physical acceleration, velocity and displacement vectors, respectively, n by 1 size; the applied loading is composed of two parts: $\mathbf{R}_0$ is the invariant spatial portion and $r(t)$ is the time varying portion.

For a modal response analysis, the physical coordinates of equation (1) are transformed to modal coordinates, q(t), by a retained set of eigenvectors of the system. Φ are eigenvectors, $\Phi=[\phi_1\ \phi_2 \cdots \phi_n]$. If the system is classically damped, $\mathbf{u}(t)=\Phi q(t)$ is used to transform equation (1) to n decoupled equations of modal coordinates.

$$\ddot{q}_i+2\beta_i\omega_i\dot{q}_i+\omega_i^2 q_i=\phi_i^T\mathbf{R}_0 r(t) \quad (2)$$

where ϕ_i, ω_i and β_i are the i th eigenvector, natural frequency of structure and damping ratio.

The mode displacement (MD) method, the standard procedure for determining responses, consist of expanding the modal responses solved from equation (2) into approximate physical responses using the retained modes[3]:

$$\mathbf{u}(t)=\sum_{i=1}^{m}\phi_i q_i(t) \qquad (m<<n) \quad (3)$$

where n is the degree of the freedom of system and m is number of retained modes. The solution of the MD method is reduced by m order and can be calculated more economically than the direct integration methods because m is much smaller than n, the degree of the freedom of the system.

If the system is non-classically damped, equation (1) can not be transformed to uncoupled equations because the damping matrix, C is not diagonalized. To apply mode superposition method to non-classically damped system, the second-order differential equation (1) can be transformed into a first-order differential equation by doubling the size of the system[4]:

$$\mathbf{B}\dot{\mathbf{y}}(t)-\mathbf{A}\mathbf{y}(t)=\mathbf{F}_0 r(t) \quad (4)$$

where A and B is the $2n$ by $2n$ matrix composed of M, C and K and y(t) is the $2n$ by 1 vector:

$$\mathbf{B}=\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix},\quad \mathbf{A}=\begin{bmatrix}-\mathbf{K} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{bmatrix},\quad \mathbf{y}(t)=\begin{Bmatrix}\mathbf{u}(t)\\ \dot{\mathbf{u}}(t)\end{Bmatrix},\quad \mathbf{F}_0=\begin{Bmatrix}\mathbf{R}_0\\ 0\end{Bmatrix} \quad (5)$$

where **0** is the n by 1, zero vector. For a modal response analysis, the physical coordinates of equation (4) are transformed to modal coordinates, z(t), by a retained set of eigenvectors of the system, Ψ.

$$\mathbf{y}(t)=\Psi z(t) \quad (6)$$

where Ψ are determined from the eigenvalue problem, $\mathbf{A}\psi_i=s_i\mathbf{B}\psi_i$. ψ_i, i th eigenvector, and s_i ,corresponding eigenvalue, may occur in complex-conjugate pairs, respectively.

For non-classically damped systems, equation (4) can be reduced to $2n$ decoupled modal equations

$$\dot{z}_i-s_i z_i=\psi_i^T\mathbf{F}_0\, r(t) \quad (7)$$

where z_i occurs in conjugate pairs

The mode displacement (MD) method for non-classically damped system can be applied to determine approximate physical responses using retained modes:

$$\mathbf{y}(t)=\sum_{i=1}^{2q}\psi_i z_i(t) \qquad (q<<n) \quad (8)$$

where n is the degree of the freedom of original system and $2q$ is number of retained modes.

2.1 MODE ACCLERATION METHOD FOR NON-CLASSICALLY DAMPED SYSTEM

For the modal response analysis, the mode superposition method generally uses the lower $2q$ modes among the total $2n$ modes as in equation (8). The truncated high modes which are not retained in the modal response analysis, induce the error of approximation.[1] The MA method and the MT method reduce the error efficiently by including the effect of truncated high modes without generating the additional modes except $2q$ modes.

But the MA and MT methods are used only for classically damped system because of the limitations of non-classical system such as difficulty of diagonalization and the inefficiency of generating the eigenvectors. In this paper, the MA and MT methods are expanded to the non-classically damped systems using state space approach. To develop the MA algorithm, equation (4) is rewritten as

$$\mathbf{y}(t) = \mathbf{A}^{-1}\mathbf{B}\dot{\mathbf{y}}(t) - \mathbf{A}^{-1}\mathbf{F}(t) \qquad (9)$$

Using equation (8), the displacements of the MA algorithm for non-classically damped system, $\mathbf{y}_{t_{ma}}(t)$, are given by :

$$\mathbf{y}_{ma}(t) = \mathbf{A}^{-1}\mathbf{B}\sum_{i=1}^{2q}\psi_i\dot{\mathbf{z}}_i - \mathbf{A}^{-1}\mathbf{F}(t) \qquad (q << n) \qquad (10)$$

The response by the MA method consists of two parts.

$$\mathbf{y}_{ma}(t) = \mathbf{y}_s(t) + \mathbf{y}_{t_{ma}}(t) \qquad (11)$$

The first portion, $\mathbf{y}_s(t)$, is the portion of the displacement solution obtained from the retained modes and so same with the displacement solution of the MD method given by equation (8). The second part, $\mathbf{y}_{t_{ma}}(t)$, is the portion of the displacement solution lost due to the modal truncation, which is given by subtracting equation (8) from equation (10) :

$$\mathbf{y}_{t_{ma}} = -\mathbf{A}^{-1}\mathbf{R}_t r(t) \qquad (12)$$

Equation (12) is the portion of the solution not represented by the modes retained in the analysis, which approximates the non-modally represented solution. In the equation (12), $\mathbf{R}_t$ is force truncation vector defined as $\mathbf{R}_t = \mathbf{F}_0 - \mathbf{R}_s$. $\mathbf{R}_s$ is the modally represented spatial load vector and given by $\mathbf{R}_s = \mathbf{B}[\ \psi_i\ \overline{\psi}_i\ ][\psi_i\ \overline{\psi}_i\ ]^T\mathbf{F}_0$. ψ_i and $\overline{\psi}_i$ are the eigenvectors in complex-conjugate pairs. Modal-physical representation of displacements and loads like $\mathbf{R}_s$ and $\mathbf{R}_t$ are well documented by J. M. Dickens, J. M. Nakagawa and M. J. Wittbrodt[5].

$\mathbf{R}_s$ for non-classically damped system is calculated from complex-conjugate pairs of eigenvector because $\mathbf{R}_s$ has physical meaning of modally represented spatial load vector that can not be in the complex form. The MA method contains both modally represented portion , $\mathbf{y}_s(t)$, and non-modally represented portion, $\mathbf{y}_{t_{ma}}(t)$.

2.2 MODAL TRUNCATION AUGMENTAION METHOD FOR NON-CLASSICALLY DAMPED SYSTEM

The response by the MT method consists of two parts as in the MA method.[5]

$$\mathbf{y}_{mt}(t) = \mathbf{y}_s(t) + \mathbf{y}_{t_{mt}}(t) \qquad (13)$$

The first portion, $\mathbf{y}_s(t)$, is the portion of the displacement solution obtained from the retained modes and so same with the displacement solution of the MD method given by equation (8) as in the MA method. The second part, $\mathbf{y}_{t_{mt}}(t)$, is the portion of the displacement solution lost due to the modal truncation. The MT method approximates the non-modally represented solution by the MT vector,P. A MT vector is determined by solving for the displacement vector, $\overline{\mathbf{P}}$,due to the force truncation vector.

$$\mathbf{A}\overline{\mathbf{P}} = \mathbf{R}_t \qquad \mathbf{P} = \frac{1}{\alpha}\overline{\mathbf{P}} \qquad (14)$$

where $\alpha = (\overline{\mathbf{P}}^T\mathbf{B}\overline{\mathbf{P}})^{1/2}$. The MT vector has a mathematically consistence with Rayleigh-Ritz approximation where the assumed Ritz basis vectors are derived using the spatial force truncation vector. The MT vector is orthogonal on the matrix **A** and **B** but do not satisfy the eigenvalue problem at each equation.

For a modal response analysis by the MT method, the transformation, $\mathbf{y}_{t_{mt}} = \mathbf{P}z_P(t)$, is used to reduce the equation (4):

$$\dot{z}_P(t) - s_P z_P(t) = \mathbf{P}^T \mathbf{R}_t r(t) \tag{15}$$

where $\mathbf{P}^T\mathbf{BP} = 1$ and $s_p = \mathbf{P}^T\mathbf{AP}$. After $z_P(t)$ is calculated in equation (15), $z_P(t)$ can then be back transformed using the equation, $\mathbf{y}_{t_{mt}} = \mathbf{P}z_P(t)$, to yield the MT solution for the displacements contained in the non-retained modes.

The MA and MT method both approximate the non-modally represented solution. In classically damped system, it would be expected that the MT method would give better results overall than the MA method because of added dynamics.[5] In non-classically damped system, however, the MT method has different characteristics from the classically damped system. The characteristics of the MT method for non-classically damped system are discussed in two points, the stability of the solution of the MT method and the characteristics of the solution by the MT method when the solutions are stable.

3. THE CHARATERISTICS OF THE MODAL TRUNCATION AUGMENTAION METHOD FOR NON-CLASSICALLY DAMPED SYSTEM

To check the stability of the MT method, the general solution of equation (15) is solved for the initial condition, $z_P(0) = 0$, when the harmonic force, $r(t) = sin(\omega t)$, is applied:

$$z_P(t) = \frac{\mathbf{P}^T\mathbf{R}_t}{s_P^2 + \omega^2}\left(\omega e^{s_P t} - s_P \sin(\omega t) - \omega \cos(\omega t)\right) \tag{16}$$

Equation (16) is stable if the $s_p = \mathbf{P}^T\mathbf{AP}$ satisfies the condition, $s_p = \mathbf{P}^T\mathbf{AP} < 0$. Therefore, the MT method is stable and applicable to modal response analysis for non-classically damped system when the condition, $s_p = \mathbf{P}^T\mathbf{AP} < 0$ is satisfied. If the condition is not satisfied, the MT solution would be divergent as time goes on. The stability of the solution by the MT method depends on the MT vector, P, because the matrix **A** is generally indefinite [6] and so depends on the force truncation vector, $\mathbf{R}_t$, because the MT vector, P, is formed from the $\mathbf{R}_t$.

When the MT solution is stable, the characteristics of the solution is presented as follows. The matrix **A** consists of mass matrix, **M**, and stiffness matrix , **K** and so the elements of the matrix **A** have the value of the material properties of the structure. On the other hand, the frequency of input loading, ω, has the value of range from several tens to several hundreds in rad/sec. Therefore, generally, the absolute value of $s_p = \mathbf{P}^T\mathbf{AP}$ is much bigger than the ω, the frequency of input loading; $|s_P| >> \omega$.

Using the condition, $|s_P| >> \omega$, Equation (16) can be simplified:

$$z_P(t) \approx \frac{-\mathbf{P}^T\mathbf{R}_t}{s_p} sin(\omega t) \tag{17}$$

Then $z_P(t)$ can be back transformed using equation, $\mathbf{y}_{t_{mt}} = \mathbf{P}z_P(t)$, to yield the MT solution for the displacements contained in the non-retained modes:

$$\mathbf{y}_{t_{mt}} = \mathbf{P}z_P(t) = -\frac{\mathbf{P}\mathbf{P}^T\mathbf{R}_t}{s_P} sin(\omega t) \tag{18}$$

Using the equation (14), the MT vector, P, can be represented as follows

$$\mathbf{P} = \frac{1}{\alpha}\mathbf{A}^{-1}\mathbf{R}_t \tag{19}$$

Substituting equation (19) into equation (18), the MT solution for the displacements has a form similar with the MA solution for the displacements given by equation (12).

$$\mathbf{y}_{t_{mt}} = -\mathbf{A}^{-1}\mathbf{R}_t \frac{\mathbf{P}^T\mathbf{R}_t}{\alpha\, s_p} sin(\omega t) \quad (20)$$

Substituting $\hat{\mathbf{R}}_t = \alpha\mathbf{AP}$ and $s_p = \mathbf{P}^T\mathbf{AP}$ into equation (20), the MT solution is simplified and same with the MA solution:

$$\mathbf{y}_{t_{mt}} = -\frac{s_P\,\mathbf{P}^T\hat{\mathbf{R}}_t}{\alpha(s_P^2+\omega^2)}\mathbf{A}^{-1}\hat{\mathbf{R}}_t\, sin(\omega t) \approx -\mathbf{A}^{-1}\hat{\mathbf{R}}_t\, sin(\omega t) = \mathbf{y}_{t_{ma}} \quad (21)$$

In summary, the applicability of the MT method for non-classically damped system is limited by the condition for stability, $s_p = \mathbf{P}^T\mathbf{AP} < 0$. When the condition is satisfied, the MT method is stable and applicable to modal response analysis for non-classically damped system. When the MT solution is stable, the result of the MT solution is same with the MA solution as proved in equation (17)-(21).

4. NUMERICAL EXAMPLES

4.1 CANTILEVER BEAM SUBJECTED TO EARTHQUAKE

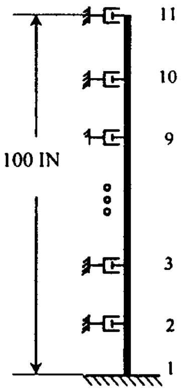

Fig. 1. Beam configuration

This numerical example shows that the MA and MT methods are more efficient than the MD method for non-classically damped system as in classically damped system. The system shown in Figure 1 is a cantilever beam with a lumped translational viscous-damper attached at each node. The cantilever beam is modeled by 10 equal finite elements deduced from elementary beam theory. The Young's modulus, area, inertia and density of beam are , respectively, 3.0×10^7 psi, 4 in^2, 1.25 in^4 and 7.41×10^{-4} lb/in^3. The damping coefficient of the tangential damper is 0.1 lb·sec/in. The load applied to systems is El-Centro earthquake. Table 1 shows the complex-conjugate pairs of eigenvalue.

The results of the response analysis are shown in Figure 2. For the comparison with accurate solution, each result is divided by the result of the direct integration method that is exact solution of the system. In the legend of the figure, md1 means that the MD method uses the lowest one eigenvector for a modal response analysis and md2 means that the MD method uses the lowest two eigenvectors and so on. The legend, ma1 and mt1, mean that the MA and MT method use the lowest one eigenvector for a modal analysis like in the MD method.

Figure 2 shows that the MA and MT solutions are more efficient than the MD solution as previously discussed and shows that the MT solution is same with the MA solution when the MT method satisfies the condition, $s_p = \mathbf{P}^T\mathbf{AP} < 0$.

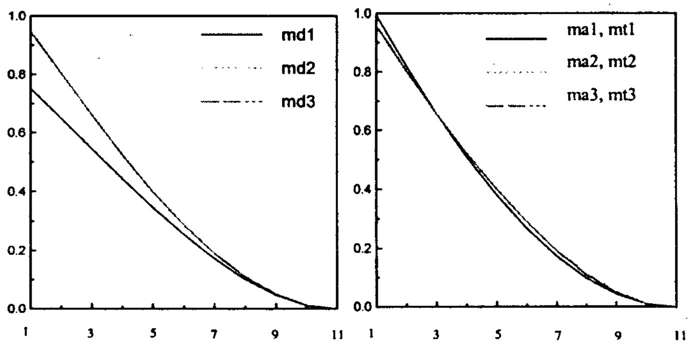

Fig. 2. Moment by each method

Table 1. Eigenvalue set

MODE NUMBER	*EIGENVALUES*
1	− 4.43482 − 39.29620i
2	− 4.43482 + 39.29620I
3	− 88.4454 − 231.3995i
4	− 88.4454 + 231.3995i
5	− 677.3535 − 147.892i
6	− 677.3535 + 147.892i

Through the above numerical example, it is shown that the expanded MA and MT methods are well applied to a non-classically damped system and the results has same trend with classically damped system in which the MA and MT methods are more efficient. In classically damped system, it would be expected that the MT method would give better results overall than the MA method because of added dynamics.[8] In non-classically damped system, however, the MT method give the same results with the MA method when the MT method is stable and so applicable.

4.2 10-STORY SHEAR BUILDING

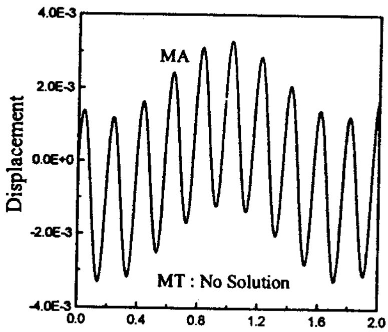

Fig. 3 Response by the MA and MT method

This numerical example shows a divergent case of the MT method. The structure is non-classically damped system, which has 10 degrees of freedom with a lumped translational viscous-damper attached at seventh floor. The responses by the MA and MT methods are calculated when harmonic load, sin(wt), is applied at first floor. In this example, the condition, $s_p = \mathbf{P}^T\mathbf{AP} < 0$, is not satisfied because of the value, $s_p = 6.2443 \times 103$.

Figure 3 shows that the MT method can not be applicable because the solution is divergent. The MA solution is , however, stable .

5. CONCLUSIONS

The MA and MT methods were expanded to the non-classically damped systems for a modal response analysis of the structures with non-classically damping. The applicability of expansion was verified by the closed form solutions and the numerical examples.

The expanded MA and MT methods are more efficient than the simple MD method because the non-modally represented portion is included in the MA and MT method as in classically damped system. The result of the expanded MA and MT method for the non-classically damped system is identical in almost case. However, considering the stability of the solution, the expanded MA method is better than the MT method because the expanded MT method are not always stable according to the condition, $s_p = \mathbf{P}^T\mathbf{AP} < 0$. Therefore the expanded MA method is better than the MT method for an analysis of the non-classically damped system.

REFERENCE

[1] G. B. Warburton and S. R. Soni, "Erros in Response Calculations for Non-Classically Damped Structures", Earthquake Engineering and Structural Dynamics, Vol. 5, 1987, pp.365-376.

[2] D. E. Newland, Mechanical Vibration Analysis and Computation, Longman Scientific & Technical, 1989.

[3] Roy R. Craig, Jr., Structural Dynamics, John Wiley&Sons, 1981.

[4] Takeru Igusa, Armen Der Kiureghian and Jerome L. Sackman, "Modal Decomposition Method for Stationary Response of Non-Classically Damped Systems.", Earthquake Engineering and Structural Dynamics, Vol.12, 1984, pp.121-136

[5] J. M. Dickens, J. M. Nakagawa and M. J. Wittbrodt, "A Critique of Mode Acceleration and Modal Truncation Augmentation Methods for Modal Response Analysis.", Computers and Structures, Vol.62, No.6, 1997, pp. 985-998

[6] A. Ibrahimbegovic, H. C. Chen, E. L. Wison and R. L. Taylor, "Ritz Method for Dynamic Analysis of Large Discrete Linear Systems with Non-proportional Damping.", Earthquake Engineering and Structural Dynamics, Vol.19, 1990, pp.877-889