Technique of Checking Missed Eigenvalues for Eigenproblem Including Damping

*Hyung-Jo Jung[1] and In-Won Lee[2]

Department of Civil Engineering, KAIST, Taejon 305-701, Korea

Sun-Kyu Park[2]

Department of Civil Engineering, SKKU, Suwon 440-746, Korea

ABSTRACT

The eigenvalue analysis including damping should be necessarily performed in the case of the nonproportionally damped structures such as the soil-structure interaction problem and the structural control problem. The most eigenvalue analysis methods such as the subspace iteration method and the Lanczos method may miss some eigenpairs in the required ones. Therefore, the eigenvalue analysis method must include the technique to check the missed eigenvalues to become the practical tools. In the case of the undamped or proportionally damped structures the missed eigenvalues can be checked by using the Sturm sequence property, while in the case of the nonproportionally damped structure a technique of checking the missed eigenvalues has not been developed yet. In this paper, the mathematical properties such as the extension of the Sturm sequence property, the Routh-Hurwitz criterion and the argument principle have been reviewed, and then a technique that can check the missed eigenvalues for the eigenproblem including damping by applying the argument principle is proposed. To show the effectiveness of the proposed method, a numerical example is considered.

INTRODUCTION

The eigenvalue analysis methods such as the subspace iteration method and the Lanczos method calculate only the lowest eigenvalues and the corresponding eigenvectors, because it usually gives a very good approximation to the exact dynamic response although the modal transformation matrix is not composed of the complete eigenvector set of the structure but the lowest small portion of this set. However, the above eigenvalue analysis methods may miss some eigenpairs in the required ones. If the modal transformation process is performed with the lowest incomplete eigenvectors, the results of the dynamic analysis may not be a good

1) Graduate Student
2) Professor

approximation to the exact response. In this case, the missed eigenpairs should be extracted to construct the lowest complete eigenvector set, before the modal transformation step is performed.

The well-known Sturm sequence property has hitherto been applied to check the missed eigenvalues. This technique is used in the commercial FEM program such as ADINA. However, this technique can only be applied to the eigenproblem without the damping matrix such as the case of the undamped and proportionally damped system. In most real systems, the damping matrix is nonproportional. In the case of the nonproportionally damped system such as the soil-structure interaction problem, the structural control problem and composite structures, the eigenproblem including damping should be analyzed for the exact dynamic response. Although the eigenvalue analysis including damping is performed well, the accuracy of the dynamic response is not guaranteed because the missed eigenvalues may exist in the required ones. To guarantee the exact dynamic response, a technique that can check the missed eigenvalues for the eigenproblem including damping is required.

To find an appropriate mathematical property that can be applied to the technique of checking the missed eigenvalues, various mathematical properties such as the extension of the Sturm sequence property, the Gerschgorin's theorem, the Routh-Hurwitz criterion and the argument principle have been reviewed comprehensively. And then a technique that can check the missed eigenvalues for the eigenproblem including damping by applying the argument principle is developed. By analyzing a numerical example, it is verified that the proposed method can check exactly the missed eigenvalues.

TECHNIQUE OF CHECKING MISSED EIGENVALUES

Argument Principle

Let $f(\lambda)$ be a polynomial of order n in λ as follows:

$$f(\lambda) = a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + a_n\lambda^n \tag{1}$$

where λ is complex numbers and $a_i\ (i = 0,1,\ldots,n)$ the real coefficients.

If $f(\lambda)$ is analytic inside and on a simple closed contour S except for a finite number of poles inside the contour S, the following equation is introduced

$$\frac{1}{2\pi i}\oint_S \frac{f'(\lambda)}{f(\lambda)} d\lambda = N - P \tag{2}$$

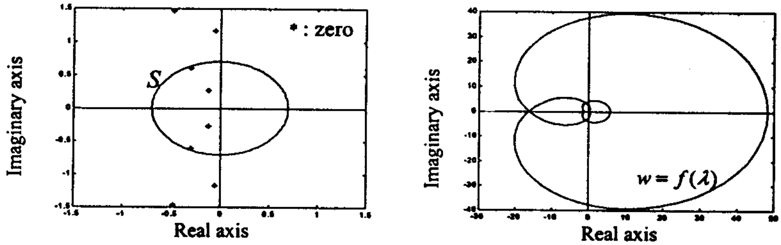
where N is the number of zeros and P the number of poles of $f(\lambda)$ inside S, respectively, and $i = \sqrt{-1}$.

If a pole of $f(\lambda)$ does not exist inside S, Eq. (2) reduces to the argument principle (Korn and Korn 1968):

$$N = \frac{\Delta_S\theta}{2\pi} \tag{3}$$

where $\Delta_S\theta$ is the variation of the argument θ of $f(\lambda)$ around the contour S.

Eq. (3) means that $w = f(\lambda)$ maps a moving point λ describing the contour S into a moving point w which encircles the origin of the w plane N times if $f(\lambda)$ has N zeros inside the contour S in the λ plane. As seen from Fig. 1, a moving point $w = f(\lambda)$ encircles the origin of the w plane four times if $f(\lambda)$ has four zeros inside the contour S in the λ plane.



(a) λ plane (b) w plane

Fig. 1 Argument principle

Proposed Method

The eigenproblem including damping can be written as:

$$\lambda_i^2 \boldsymbol{M} \boldsymbol{\phi}_i + \lambda_i \boldsymbol{C} \boldsymbol{\phi}_i + \boldsymbol{K} \boldsymbol{\phi}_i = 0 \quad (i = 1,2,\dots,n) \tag{4}$$

where λ_i is the ith eigenvalue and $\boldsymbol{\phi}_i$ the corresponding eigenvector.

Let us consider the large structural system of order n. By using the symbolic algebraic operations as in the previous section, it is impossible to find the characteristic polynomial $f(\lambda_i)$ of the quadratic eigenproblem as Eq. (4):

$$f(\lambda_i) = \det(\lambda_i^2 \boldsymbol{M} + \lambda_i \boldsymbol{C} + \boldsymbol{K}) = a_0 + a_1 \lambda_i + \cdots + a_{2n-1} \lambda_i^{2n-1} + a_{2n} \lambda_i^{2n} \tag{5}$$

where λ_i is a complex number on the simple closed contour S. The inside of the contour S means the domain that is assigned to check the missed eigenvalues. Since the analytical solution in the field of the symbolic computation cannot be calculated, the numerical solution in the field of the complex arithmetic computation should be considered.

To develop a technique of checking the missed eigenvalues using the argument principle, the discretization of the simple closed contour S and the relationship between the characteristic polynomial and the factorized matrices by the $\boldsymbol{LDL}^T$ factorization should be considered. First, the simple closed contour S in Fig. 1(a) is considered as the set of the checking points as described in Fig. 2. And then, the $\boldsymbol{LDL}^T$ factorization process is performed at each checking point. Then, the argument θ_j at the checking point j can calculate as follows:

$$f(\lambda_j) = \det(\lambda_j^2 \boldsymbol{M} + \lambda_j \boldsymbol{C} + \boldsymbol{K}) = \det \boldsymbol{L} \boldsymbol{D} \boldsymbol{L}^T = \prod_{i=1}^{n} d_{ii} = r_j \angle \theta_j \tag{6}$$

where d_{ii} is the diagonal elements of the diagonal matrix $\boldsymbol{D}$, and r_j and θ_j the magnitude and argument of the value $f(\lambda_j)$ in polar form, respectively.

The number of the eigenvalues inside the contour S is calculated by summing the variation of the argument of each checking point as follows:

$$N = \frac{\sum \Delta_S \theta_j}{2\pi} \tag{7}$$

where $\sum \Delta_S \theta_j$ is the total variation of the argument, that is, the sum of the variation of the argument at each checking point. An eigenvalue of multiplicity k is counted k times (Franklin et al. 1994). Hence, the technique of checking the missed eigenvalues using the argument principle can be applied to an eigenvalue analysis of a structure with multiple eigenvalues.

The graphical representation of the proposed method is described in Fig. 2. Since the eigenvalues are always complex conjugate pairs in the case of the eigenproblem of the underdamped system, the simple closed contour S is only considered in the second quarter-plane as in Fig. 2. Hence, if the number of the calculated eigenvalues by performing an eigenvalue analysis is q, then the number of the eigenvalues considered for the checking process (p) is the half of the number of the above calculated eigenvalues (i.e., $p = q/2$). By extensive experience, we know that the appropriate number of the initial checking points should be at least $6p$ to check completely without any missed eigenvalues.

The process for checking the missed eigenvalues using the argument principle can be briefly described as follows. First, one checks from the origin to the maximum magnitude ($\rho = 1.001|\lambda_q|$) along the imaginary axis (① in Fig. 2), and the second quarter-plane along the quadrant with radius ρ (② in Fig. 2). In each process, one selects about $3p$ checking points, and then one performs the LDL^T factorizing process at each checking point. The checking process in the real axis can skip because of no variation of the argument (③ in Fig. 2). If the decrease and the aggressive variation (about 200°) of the argument between two adjacent checking points may occur, the extra checking points between two adjacent checking points should be added. The total variation of the arguments is calculated by summing the argument of each checking point. Finally, it is checked the missed eigenvalues by comparing the total rotation number (N in Eq. (7)) with the number of the considered eigenvalues (p).

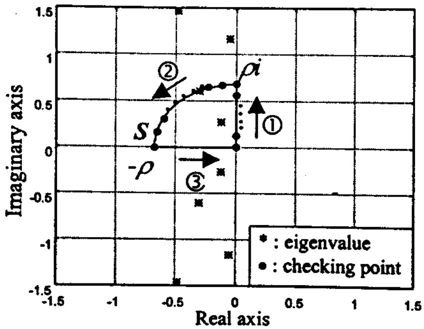


Fig. 2 Process for checking the missed eigenvalues using the argument principle

NUMERICAL EXAMPLE

A simple spring-mass-damper system that has the correct analytical solutions is considered to verify that the proposed method can check exactly the missed eigenvalues for the eigenproblem including damping. The finite element discretization of the system results in a diagonal mass matrix, a tridiagonal damping and stiffness matrices of the following forms (Chen and Taylor 1988)

$$M = mI;\ C = \alpha M + \beta K;\ K = k\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \qquad (8, 9, 10)$$

where α and β are the damping coefficients of the Rayleigh damping. The analytical solutions can be resulted through following relationships

$$\lambda_{2i-1,2i} = -\xi_i\omega_i \pm j\omega_i\sqrt{1-\xi_i^2} \quad \text{for } i = 1,\ldots,n \qquad (11)$$

$$\xi_i = \frac{1}{2}\left(\frac{\alpha}{\omega_i} + \beta\omega_i\right) \text{ and } \omega_i = 2\sqrt{\frac{m}{k}}\sin\frac{2i-1}{2n+1}\frac{\pi}{2} \qquad (12, 13)$$

where ω_i and ξ_i are the undamped natural frequency and modal damping ratio, respectively.

A system with order 50 is used in analysis. k and m are 1, and the coefficients, α and β, of the Rayleigh damping are 0.05 and 0.5, respectively. The lowest six eigenvalues by analytical solutions are expressed as in Table 1. No missed eigenvalues exist in the lowest six eigenvalues because these are the analytical solutions.

Table 1 The lowest six eigenvalues

Mode Number	Eigenvalues	
	Real	Imaginary
1	−0.02524	+0.01817
2	−0.02524	−0.01817
3	−0.02718	+0.08923
4	−0.02718	−0.08923
5	−0.03103	+0.15224
6	−0.03103	−0.15224

To verify that the proposed method can check exactly the missed eigenvalues for the eigenproblem including damping, the checking process is performed. First, since the second quarter-plane in the complex plane is only considered, the number of the considered eigenpairs is three ($p=3$). And, the maximum magnitude is calculated by 1.001 times the absolute value of the sixth eigenvalue ($\rho = 1.001|\lambda_6| = 0.1555$). To select the initial checking points, the line from the origin to ρi on the imaginary axis and the quadrant with radius ρ in the second quarter-plane are divided into nine equal parts, respectively. Then, the factorizing process at each checking point is performed, and the variation of the argument at each one is reviewed. After the additional checking points are selected through the review on the results of the first checking process, the second and final checking process is performed. The results of each factorizing process are described in Table 2 and Fig. 3.

Table 2 The arguments and the variations of the arguments

Initial checking process				Second checking process				Final checking process				
λ_j	θ_j	$\Delta_s\theta_j$	Y/N	λ_j	θ_j	$\Delta_s\theta_j$	Y/N	λ_j	θ_j	$\Delta_s\theta_j$	Y/N	$\sum\Delta_s\theta_j$
origin	0.00	-	-									0.00
$\rho/9$	89.1	89.1	N									89.1
$2\rho/9$	172.1	83.0	N									172.1
$3\rho/9$	239.9	67.8	N									239.9
$4\rho/9$	305.1	65.2	N									305.1
$5\rho/9$	19.7	74.6	N									379.7
$6\rho/9$	98.1	78.4	N									458.1
$7\rho/9$	168.0	69.9	N									528.0
$8\rho/9$	237.9	69.9	N									597.9
ρ	314.6	76.7	N									674.6
$\rho\angle 100°$	319.6	5.0	N									679.6
								$\rho\angle 100.5°$	321.0	1.4	N	681.0
								$\rho\angle 101.0°$	322.2	1.2	N	682.2
								$\rho\angle 101.5°$	24.9	62.7	N	744.9
				$\rho\angle 102°$	**132.8**	**173.2**	Y	$\rho\angle 102°$	132.8	107.9	N	852.8
				$\rho\angle 104°$	137.3	4.5	N					857.3
				$\rho\angle 106°$	138.1	0.8	N					858.1
				$\rho\angle 108°$	139.1	1.0	N					859.1
$\rho\angle 110°$	**140.5**	**180.9**	Y	$\rho\angle 110°$	140.5	1.4	N					860.5
$\rho\angle 120°$	152.6	12.1	N									872.6
$\rho\angle 130°$	174.0	21.4	N									894.0
$\rho\angle 140°$	202.1	28.1	N									922.1
$\rho\angle 150°$	236.1	34.0	N									956.1
$\rho\angle 160°$	274.7	38.6	N									994.7
$\rho\angle 170°$	316.5	41.8	N									1,016.5
$\rho\angle 180°$	0.0	43.5	N									1,080.0

where $0° \le \theta_j < 360°$, $\Delta_s\theta_j = \theta_j - \theta_{j-1}$, and '*Y*' means that the additional checking points are required and '*N*' means that the additional ones are not required.

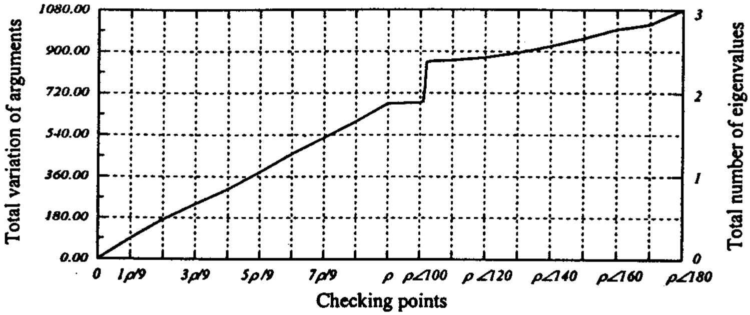


Fig. 3 Change of total variation of arguments and total no. of eigenvalues with checking points

Since the total variation of the argument is 1,080°, the total number of rotations is as follows:

$$N = \frac{\sum \Delta_S \theta_j}{2\pi} = \frac{1{,}080^\circ}{360^\circ} = 3\,.$$

Finally, the missed eigenvalues are checked by comparing the number of rotations with the number of the considered eigenvalues. Since the number of rotations and the number of the considered eigenvalues are all the same (i.e., $N = p = 3$), the missed eigenvalues do not exist in the simple closed contour S. From this result, it is verified that the proposed method can check exactly the missed eigenvalues for the eigenproblem including damping.

CONCLUSIONS

This paper proposes a technique of checking the missed eigenvalues for the eigenproblem including damping. Through a comprehensive review on various mathematical properties, it has been verified that the argument principle is an appropriate property in the case of the eigenproblem including damping. Using the argument principle, a technique of checking the missed eigenvalues for the eigenproblem including damping is developed. The effectiveness of the proposed method is verified by analyzing a simple spring-mass-damper system.

The proposed method needs relatively a large number of operation counts compared with the technique using the Sturm sequence property. This cannot be inevitable, because the proposed method is executed in the complex plane whereas the Sturm sequence check is performed on the real axis. To more effectively apply the proposed method to the practical problem, the research to reduce the operation counts should be performed.

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