A NUMERICALLY STABLE
SUBSPACE ITERATION METHOD WITH SHIFT

H.-J. Jung*, J.-W. Oh† and I.-W. Lee*
*Department of Civil Engineering, KAIST, Taejon, Korea
†Department of Civil and Environmental Engineering, Hannam University, Taejon, Korea

Abstract

A numerically stable eigensolution method for structures is presented. The proposed method is developed by improving the well-known subspace iteration method with shift. A major difficulty of the subspace iteration method with shift is that because of singularity problem, a shift close to an eigenvalue cannot be used, resulting in slower convergence. In this paper, the above singularity problem has been solved by introducing side conditions without sacrifice of convergence. The proposed method is always nonsingular even if a shift is on a distinct eigenvalue or multiple ones. This is one of the significant characteristics of the proposed method. The nonsingularity is proved analytically. The convergence of the proposed method is at least equal to that of the subspace iteration method with shift, and the operation counts of the above two methods are almost the same when a large number of eigenpairs are required. To show the effectiveness of the proposed method, two numerical examples are considered.

1 Introduction

The eigensolution method is very important in a dynamic analysis of structures when the mode superposition method is used. Many eigensolution methods have been developed, and among these methods, the subspace iteration method has hitherto been known to be very efficient, and so has been widely used.

The subspace iteration method was developed and named by Bathe [1-2]. This method combines the simultaneous inverse iteration method and the Rayleigh-Ritz analysis. The following shortcomings have been identified after extensive use of the method [3]. These include: (1) slow convergence and large computational and storage costs when a relatively large number of eigenpairs are required; (2) significantly computational effort required to form and solve the subspace eigenproblem when a large number of eigenpairs are required; and (3) missed eigenvectors caused by a poor choice of starting trial vectors.

To overcome the above shortcomings, many researchers have studied a variety of acceleration procedures of the subspace iteration method. The techniques employed include Chebyshev polynomials [4], over-relaxation method [5-6], shifting technique [7], exploitation by partitioning a large structure into a number of substructures [8], improving the selection initial vectors [9], selective repeated inverse iteration and multiple inverse iteration [10-11], and subspace iteration by omitting some of the Rayleigh-Ritz procedure from certain iteration steps [12-13].

Among the above accelerated techniques, a shifting technique is well-known and effectively used in the commercial FEM programs such as ADINA [14]. However, since the singularity may occur during the use of the shifting technique in the accelerated scheme such as the subspace iteration method with shift, the shift must be carefully chosen to avoid the singularity. It is a significant disadvantage of the subspace iteration method with shift.

Jung, Kim and Lee [15] have developed a method that always guarantees the numerical stability and maintains the convergence rate of the subspace iteration method with shift even if a shift is an exact eigenvalue itself. However, the method can only be applied to the structures with distinct natural frequencies. If a structure with multiple natural frequencies is analyzed by the method, the singularity problem may still occur.

In this paper, when the eigenvalue analysis for a structure with multiple eigenvalues is performed, an eigensolution technique that always guarantees the numerical stability is developed by improving the method of Jung, Kim and Lee [15]. That is, the proposed method is always numerically stable even if a shift is on a distinct eigenvalue or multiple ones.

The subspace iteration method with shift is briefly reviewed in the next chapter. Chapter Three includes the theory, the proof of the numerical stability and the operation counts of the proposed method. The effectiveness of the proposed method is verified by the results of numerical examples in Chapter Four. Chapter Five is the concluding remarks.

2 Subspace Iteration Method with Shift

The eigenproblem of the structural dynamics may be written as follows [16]:

\[ KX = MX\Lambda \]  

where \( K \) and \( M \) are the stiffness and mass matrices of the structure of the order \( n \) respectively, the columns of \( X \) the eigenvectors, and \( \Lambda \) a diagonal matrix with eigenvalues on its diagonal.
Applying a shift $\mu$ to Equation (1) gives

$$(K - \mu M)X = M X \Omega$$  \hspace{1cm} (2)$$

where

$$\Omega = \Lambda - \mu I.$$  \hspace{1cm} (3)

and $I$ is the unit matrix.

Suppose that the $p$ smallest eigenvalues $\lambda_i (i = 1, 2, \ldots, p)$ and corresponding eigenvectors $x_i$ are required. For faster convergence, $q$ trial vectors are normally used with $q = \min\{2p, p + 8\}$. Then, the algorithm of the subspace iteration method with the shift $\mu$ can be described as follows:

**Step 1.** Find the eigenvector approximations $\overline{X}^{(k+1)}$ by the simultaneous inverse iteration method;

$$(K - \mu M) \overline{X}^{(k+1)} = M X^{(k)}$$  \hspace{1cm} (4)$$

where $\overline{X}^{(k+1)}$ and $X^{(k)}$ are the $(n \times q)$ matrices.

**Step 2.** Compute the projections of the matrices $(K - \mu M)$ and $M$;

$$\overline{K}^{(k+1)} = \overline{X}^{(k+1)}^T (K - \mu M) \overline{X}^{(k+1)}$$  \hspace{1cm} (5)$$

$$\overline{M}^{(k+1)} = \overline{X}^{(k+1)}^T M \overline{X}^{(k+1)}$$  \hspace{1cm} (6)$$

where $\overline{K}^{(k+1)}$ and $\overline{M}^{(k+1)}$ are the $(q \times q)$ symmetric matrices.

**Step 3.** Solve the eigenproblem of the reduced order $q$;

$$\overline{K}^{(k+1)} \overline{Q}^{(k+1)} = \overline{M}^{(k+1)} \overline{Q}^{(k+1)} \Omega^{(k+1)}$$  \hspace{1cm} (7)$$

where $\overline{Q}^{(k+1)}$ and $\Omega^{(k+1)}$ are the $(q \times q)$ matrices.

**Step 4.** Find the improved eigenvectors $X^{(k+1)}$;

$$X^{(k+1)} = \overline{X}^{(k+1)} \overline{Q}^{(k+1)},$$  \hspace{1cm} (8)$$

and the improved eigenvalues $\Lambda^{(k+1)}$;

$$\Lambda^{(k+1)} = \Omega^{(k+1)} + \mu I.$$  \hspace{1cm} (9)$$

$\Lambda^{(k+1)}$ converges to $\Lambda$ and $X^{(k+1)}$ converges to $X$ as $k$ approaches infinity. The convergence rate of the subspace iteration method with shift is

$$\frac{\lambda_j - \mu}{\lambda_{q+1} - \mu}.$$  \hspace{1cm} (10)$$

If a shift is an eigenvalue itself or very close to it, the iteration procedure becomes unstable because of the singularity problem occurring during the $LDL^T$ factorization process of the coefficient matrix. To avoid this singularity problem, that is, to guarantee the numerical ability of the subspace iteration method with shift, the following condition was adopted in the subspace iteration method [7];

$$1.01 \overline{\lambda}_{s-1} \leq \mu \leq 0.99 \overline{\lambda}_s$$  \hspace{1cm} (11)$$

where $\overline{\lambda}_{s-1}$ is the calculated approximation to the $(s-1)$th eigenvalue and $\overline{\lambda}_s$ the $s$th eigenvalue.

It means that a shift must be within a limited region resulting in slow convergence. Moreover, if the calculated approximation to an eigenvalue slightly differs from it, an eigenvalue may be inside the limited region. Then, the singularity may occur although a shift is inside the limited region. These are the significant disadvantages of the subspace iteration method with shift. The purpose of this paper is to remove the limitation in Equation (11) for choosing the value of a shift $\mu$.

### 3 Proposed Method

#### 3.1 Theory

Consider the simultaneous inverse iteration step in the subspace iteration method with shift;

$$(K - \mu M) \overline{X}^{(k+1)} = M X^{(k)}.$$  \hspace{1cm} (12)$$

If a shift $\mu$ is very close to an eigenvalue, the singularity problem occurs during the $LDL^T$ factorization process of the coefficient matrix $(K - \mu M)$ in Equation (12). Then, the $(k+1)$th eigenvector approximations $\overline{X}^{(k+1)}$ cannot be acquired, and so the iteration procedure cannot be performed any more. This is a significant disadvantage of the subspace iteration method with shift.

Jung, Kim and Lee [15] proposed the numerically stable eigensolution method. However, the method can only be applied to the structures with distinct eigenvalues. If structures with multiple eigenvalues are analyzed by the method, the singularity problem may still occur.

In this paper, to solve the above singularity problem that may occur in the case of structures with multiple eigenvalues the following procedures are proposed. First, let us consider a shift close to multiple eigenvalues. To simplify the notation in this discussion, assume that the multiplicity of the lowest eigenvalue is $s$, that is, $\lambda_1 = \lambda_2 = \cdots = \lambda_s$. Then, the inverse iteration step on the multiple eigenvalues can be expressed as follows:

$$(K - \mu M) \overline{X}^{(k+1)} = M X^{(k)} D^{(k+1)}$$  \hspace{1cm} (13)$$

where the $(n \times s)$ matrices $X^{(k)} = [x_1^{(k)}, x_2^{(k)}, \ldots, x_s^{(k)}]$, $\overline{X}^{(k+1)} = [\overline{x}_1^{(k+1)}, \overline{x}_2^{(k+1)}, \ldots, \overline{x}_s^{(k+1)}]$, the $(s \times s)$ matrix $D^{(k+1)} = \text{diag}(d_1^{(k+1)}, d_2^{(k+1)}, \ldots, d_s^{(k+1)})$ and the scalar controls the length of the vector $\overline{x}_i^{(k+1)}$.

Because there are only $(n \times s)$ Equations with $((n+1) \times s)$ unknowns, $(n \times s)$ components of $\overline{X}^{(k+1)}$ and $s$ components of $d_i^{(k+1)}$, in Equation (13), $s$ side conditions
must be introduced for the solution of Equation (13). These conditions are that the current vector set \( X_s^{(k)} \) is orthogonal to the incremental vector set \( \Delta X_s^{(k)} \) with respect to \( M \); that is,
\[
X_s^{(k)^T} M \Delta X_s^{(k)} = 0. \tag{14}
\]

Adding the mass orthornormality relation, \( X_s^{(k)^T} M X_s^{(k)} = I_s \), to the side conditions, Equation (14) yields
\[
X_s^{(k)^T} M \overline{X}_s^{(k+1)} = I_s, \tag{15}
\]
where
\[
\overline{X}_s^{(k+1)} = X_s^{(k)} + \Delta X_s^{(k)}. \tag{16}
\]

The inverse iteration step on the other eigenvalues makes use of Equation (12); that is,
\[
(K - \mu M) \overline{X}_s^{(k+1)} = M X_s^{(k)} \tag{17}
\]
where
\[
X_s^{(k)} = [x_s^{(k)}(1), x_s^{(k)}(2), \ldots, x_s^{(k)}(q)]. \tag{18}
\]

Writing Equations (13), (15) and (17) in matrix form gives
\[
\begin{bmatrix}
K - \mu M & MX_s^{(k)} \\
X_s^{(k)^T} M & 0
\end{bmatrix}
\begin{bmatrix}
\overline{X}_s^{(k+1)} \\
D_s^{(k+1)}
\end{bmatrix}
= \begin{bmatrix}
MX_s^{(k)} \\
E
\end{bmatrix} \tag{19}
\]
where the unknown \((s \times q)\) matrix \(D_s^{(k+1)} = [D_s^{(k+1)}; 0, \ldots, 0]\) and the \((s \times q)\) matrix \(E = [I_s; 0, \ldots, 0]\).

Note that \( \overline{X}_s^{(k+1)} \) from Equation (19) is used for \( \overline{X}_s^{(k+1)} \) in Equations (5) and (6) instead of \( \overline{X}_s^{(k+1)} \) in Equation (4). Equation (19) is the main linear algebraic Equation used in the proposed method.

The coefficient matrix of Equation (19) is of the order \((n + s)\), symmetric, and nonsingular. The nonsingularity is one of the significant advantages of the proposed method and will be shown in the next section.

The proposed method can be applied to practical problems as follows. After assuming that a shift is very close to a distinct eigenvalue, one performs the factorizing process of the coefficient matrix with one side condition. If the shift is not very close to a distinct eigenvalue, but multiple ones (multiplicity = \(m\)), the \((n-m)\)th pivot element in the factorizing process of the coefficient matrix usually becomes small compared with its original value and the coefficient matrix becomes singular. To avoid the singularity, the \((m-1)\) side conditions are added, and then the factorizing process of the coefficient matrix is continued. Since the storage scheme of the proposed method is the skyline algorithm, the extra operation number due to the \((m-1)\) additional side conditions is small compared with the total operation number of the factorizing process of the coefficient matrix.

### 3.2 Proof of Nonsingularity of Coefficient matrix [17-19]

The most remarkable characteristic of the proposed method is that nonsingularity is always guaranteed. Let the coefficient matrix of Equation (19) be denoted by \(C\), that is
\[
C = \begin{bmatrix}
K - \mu M & MX_s^{(k)} \\
X_s^{(k)^T} M & 0
\end{bmatrix}. \tag{20}
\]

If \(C\) is nonsingular when the shift \(\mu\) becomes multiple eigenvalues, that is, \(\mu \equiv \lambda_s = \cdots = \lambda_s\), then it will be also nonsingular for a non-close shift. The resulting \(C^\ast\) will be
\[
C^\ast = \begin{bmatrix}
K - \lambda_s M & MX_s^{(k)} \\
X_s^{(k)^T} M & 0
\end{bmatrix}. \tag{21}
\]

Nonsingularity of the proposed method is, therefore, proved by introducing the new eigenvalue problem of the resulting matrix such as
\[
C^\ast Y = M^\ast YD \tag{22}
\]
where \(D\) and \(Y\) are the eigenvalue and the associate eigenvector matrices of the new eigenvalue problem, respectively, and
\[
M^\ast = \begin{bmatrix}
M & 0 \\
0 & I_s
\end{bmatrix}, \tag{23}
\]
\[
Y = [y_1, y_2, \ldots, y_{n+s}], \tag{24}
\]
\[
D = diag(\gamma_1, \gamma_2, \ldots, \gamma_{n+s}). \tag{25}
\]

The eigenpairs of the new eigenvalue problem as Equation (22), \(y_i\) and \(\gamma_i\) for \(i = 1, 2, \ldots, n + s\), are as follows:

- **Eigenvector, \(y_i\):**
  \[
  \begin{cases}
    x_i, & e_i, \\
    e_i, & -e_i
  \end{cases}
  \begin{bmatrix}
    x_i \\
    e_i
  \end{bmatrix}
  \begin{bmatrix}
    1, & \ldots, & 1 \\
    1, & \ldots, & 1
  \end{bmatrix}
  \begin{bmatrix}
    1, & \ldots, & 1 \\
    1, & \ldots, & 1
  \end{bmatrix}
  \begin{bmatrix}
    i = 1, \ldots, s \\
    k = s+1, \ldots, n
  \end{bmatrix} \tag{26}
  \]

- **Eigenvalue, \(\gamma_i\):**
  \[
  \begin{cases}
    \lambda_k - \lambda_s & (n - s) \\
    -1, & \ldots, & -1
  \end{cases}
  \begin{bmatrix}
    \lambda_k - \lambda_s \\
    -1, & \ldots, & -1
  \end{bmatrix}
  \begin{bmatrix}
    1, & \ldots, & 1 \\
    1, & \ldots, & 1
  \end{bmatrix}
  \begin{bmatrix}
    k = s+1, \ldots, n
  \end{bmatrix} \tag{27}
  \]

where \(\lambda_i\) and \(x_i\) are the eigenvalues and eigenvectors of the system \(K X = M X \Lambda\), and \(e_i\) is the \((s \times 1)\) vector that all elements are zero except for the \(i\)th element with unity.

Considering the determinant of Equation (22), the relationship can be obtained as follows:
\[ \det(C^*) = \det(M^*) \det(D) \]
\[ = (-1)^s \det(M) \prod_{k=s+1}^{n} (\lambda_k - \lambda_s). \]  
(28)

The determinant of \( C^* \) is not zero because of \( \det(M) \neq 0 \) by definition. The nonsingularity of the coefficient matrix in Equation (19) is shown. That is, the numerical stability of the proposed method is proved analytically. The proposed method, therefore, has an advantage over the subspace iteration method with shift in that no limited regions are needed in the former.

### 3.3 Operation Counts

Let one operation equal to one multiplication that is nearly always followed by an addition. Assume that the half-bandwidths of \( K \) and \( M \) are \( m_K \) and \( m_M \). respectively.

The number of operations of the subspace iteration method with shift is \( T_s qn(2m_K + 4m_M + 2q + 4) + n(m_K^2 + 5m_K + 2m_M + 2) \), and that for the proposed method \( T_p qn(2m_K + 4m_M + 2q + 4 + s) + sn(m_K + (s + 1)/2) + n(m_K^2 + 5m_K + 2m_M + 2) \).

The symbol \( s \) is the multiplicity of the multiple eigenvalues that is on or very close to a shift. The proposed method needs more operations per iteration step, \( sn(q + m_K + (s + 1)/2) \), than the subspace iteration method with shift. Assume that the ratio is composed of the operation counts per iteration of the proposed method \( (N_p) \), that of the subspace iteration method with shift \( (N_s) \), and the difference of the operation counts per iteration for the above two methods \( (N_p - N_s) \) as follows:

\[ \text{ratio} = \frac{N_p - N_s}{N_p} \]

\[ = \frac{sn(q + m_K + (s + 1)/2)}{qn(2m_K + 4m_M + 2q + 4 + s) + sn(m_K + (s + 1)/2)} \]

Then, if the half-bandwidth of the stiffness matrix \( (m_K) \) is equal to that of the mass matrix \( (m_M) \), the above ratio can be approximated as follows:

\[ \text{ratio} \approx \frac{s}{6q}. \]  
(30)

This ratio means that the larger the number of the required eigenpairs, the smaller is the difference of the operation counts between the proposed method and the subspace iteration method with shift. That is, the number of operations for the aforementioned two methods, the subspace iteration method with shift and the proposed method, is almost the same when the number of eigenpairs to be required is large.

### 4 Numerical Examples

The three-dimensional frame structure with distinct eigenvalues and the simply supported square plate with multiple eigenvalues are analyzed to verify the effectiveness of the proposed method. The solution time spent for the first ten eigenpairs and the convergence of the proposed method are compared with those of the subspace iteration method with shift which is not used in the limited region (see Equation (11)). Each method stopped when the error norms are reduced by a factor of \( 10^5 \), which yields a stable eigensolution and sufficient accuracy in the calculated eigenpairs for practical analysis [7]. The error norm \( \varepsilon_i^{(k)} \) is defined as

\[ \varepsilon_i^{(k)} = \frac{\| (K - \lambda_i^{(k)} M) x_i^{(k)} \|_2}{\| K x_i^{(k)} \|_2}. \]  
(31)

All runs are executed in the IRIS4D-20-S17 with 10 Mips and 0.9 MFlops.

#### 4.1 Three-Dimensional Frame Structure

The geometric configuration and the material properties of the three-dimensional frame structure are shown in Figure 1. The structure discretized using 100 beam elements resulting in system of dynamic Equations with a total of 468 degrees of freedom. The consistent mass matrix is used for \( M \). The lowest ten eigenvalues of this structure are well separated as seen from Table 1.

![Figure 1: Three-dimensional framed structure](image)

<table>
<thead>
<tr>
<th>Elevation</th>
<th>Plan</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column in Front Building</td>
<td>( A = 0.2787 \ m^2, I = 8.631 \times 10^{-3} \ m^4 )</td>
</tr>
<tr>
<td>Column in Rear Building</td>
<td>( A = 0.3716 \ m^2, I = 10.787 \times 10^{-3} \ m^4 )</td>
</tr>
<tr>
<td>All Beams into x - Direction</td>
<td>( A = 0.6096 \ m^2, I = 6.6473 \times 10^{-1} \ m^4 )</td>
</tr>
<tr>
<td>All Beams into y - Direction</td>
<td>( A = 0.2787 \ m^2, I = 8.631 \times 10^{-3} \ m^4 )</td>
</tr>
<tr>
<td>( E = 2.068 \times 10^{10} \ Pa, \rho = 5.154 \times 10^2 \ \text{kg/m}^3 )</td>
<td></td>
</tr>
</tbody>
</table>

Figure 1. Three-dimensional framed structure

Some results are shown in Table 2 and in Figures 2 to 5. The solution time for two methods are summarized in Table 2. When a shift is on \( \lambda_s \), the subspace iteration method with shift and the proposed method calculate the required ten eigenpairs. However, when the shift is on \( \lambda_s \), the subspace iteration method with shift does not find the solutions while the proposed method obtains the solutions. It shows that the iteration procedure for the proposed method can converge without any singularity even if the shift is the same exactly to an eigenvalue. This is one of the significant advantages of the proposed method.
4.2 Simply Supported Square Plate

The second example is the simply supported square plate. Figure 8 shows the geometric configuration and material properties. The structure is discretized by using 36 shell elements (nine node per element) resulting in a system of dynamic Equations with a total of 701 degrees of freedom. The consistent mass matrix is used for \( M \). The lowest ten eigenvalues of the model are shown in Table 3. The eigenvalues of the model are distinct root or multiple ones.

Some results are shown in Table 4 and in Figures 9 to 14. The solution time for the two methods are summarized in Table 4. When a shift is on \( 1.01 \lambda_2 \), the subspace iteration method with shift and the proposed method obtain the required ten eigenpairs. However, when the shift is on \( 1.0001 \lambda_2 \) or on \( \lambda_2 \), the subspace iteration method with shift does not calculate the solutions while the proposed method finds the solutions. It shows that the iteration procedure for the proposed method can converge without any singularity even if the shift is exactly the same as the multiple eigenvalues.

Table 3. The lowest ten eigenvalues of the simply supported square plate

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4435E+01</td>
</tr>
<tr>
<td>2</td>
<td>0.2914E+02</td>
</tr>
<tr>
<td>3</td>
<td>0.2914E+02</td>
</tr>
<tr>
<td>4</td>
<td>0.7367E+02</td>
</tr>
<tr>
<td>5</td>
<td>0.1305E+03</td>
</tr>
<tr>
<td>6</td>
<td>0.1305E+03</td>
</tr>
<tr>
<td>7</td>
<td>0.2087E+03</td>
</tr>
<tr>
<td>8</td>
<td>0.2087E+03</td>
</tr>
<tr>
<td>9</td>
<td>0.4010E+03</td>
</tr>
<tr>
<td>10</td>
<td>0.4418E+03</td>
</tr>
</tbody>
</table>

Table 4. Solution time for the lowest ten eigenpairs of the simply supported square plate, seconds (ratio)

<table>
<thead>
<tr>
<th>Analysis methods</th>
<th>Shift ( = 1.01 \lambda_2 )</th>
<th>Shift ( = 1.00001 \lambda_2 )</th>
<th>Shift ( = \lambda_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subspace iteration method with shift</td>
<td>720.95</td>
<td>No solution</td>
<td>No solution</td>
</tr>
<tr>
<td>Proposed method</td>
<td>751.05 (1.04)</td>
<td>750.64</td>
<td>751.25</td>
</tr>
</tbody>
</table>

For each solution method, the convergence of each eigenpair is depicted in Figures 9 to 14. Figures 9 and 10 show that when the shift is on \( 1.01 \lambda_2 \) the convergence of the proposed method is nearly equal to that of the subspace iteration method with shift. Figures 11 and 12 show that when the shift is on \( 1.00001 \lambda_2 \) the proposed method converges well without any singularity while the subspace...
iteration method with shift can not converge due to the singularity. Figures 13 and 14 show that when the shift is exactly the same as the second eigenvalue, the proposed method only converges well without any singularity.

Figure 9. Error norm versus iteration number the second eigenpair of the simply supported square plate in case of shift = 1.01 \( \lambda_2 \)

Figure 10. Error norm versus iteration number the tenth eigenpair of the simply supported square plate in case of shift = 1.01 \( \lambda_2 \)

Figure 11. Error norm versus iteration number the second eigenpair of the simply supported square plate in case of shift = 1.00001 \( \lambda_2 \)

Figure 12. Error norm versus iteration number the tenth eigenpair of the simply supported square plate in case of shift = 1.00001 \( \lambda_2 \)

Figure 13. Error norm versus iteration number the second eigenpair of the simply supported square plate in case of shift = \( \lambda_2 \)

Figure 14. Error norm versus iteration number the tenth eigenpair of the simply supported square plate in case of shift = \( \lambda_2 \)
5 Conclusions

A numerically stable technique using side conditions for improving the subspace iteration method with shift has been presented. The characteristics of the proposed method identified by the analytical and the numerical results from numerical examples are summarized as follows:

(1) The nonsingularity of the proposed method is always guaranteed, which is proved analytically; even if the shift is on a distinct eigenvalue or multiple ones, the proposed method can obtain the solutions without any singularity.

(2) The convergence rate of the proposed method is at least equal to that of the subspace iteration method with shift, and the operation counts of the proposed method and the subspace iteration method with shift are almost the same when the number of eigenpairs to be required is large.

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