

Simplified algebraic method for computing eigenpair sensitivities of damped systems

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ABSTRACT: A simplified method for the eigenpair sensitivities of damped systems is presented. This approach employs a reduced equation to determine the sensitivities of eigenpairs of the damped vibratory systems with distinct eigenvalues. The derivatives of eigenpairs are obtained by solving an algebraic equation with a symmetric coefficient matrix of $(n+1)$ by $(n+1)$ dimension where n is the number of degree of freedom. This is an improved method of the previous work of Lee and Jung. Two equations are used to find eigenvalue derivatives and eigenvector derivatives in their paper. A significant advantage of this approach over Lee and Jung is that one algebraic equation newly developed is enough to compute such eigenvalue derivatives and eigenvector derivatives. Simulation results indicate that the new method is highly efficient in determining the sensitivities of eigenpairs of the damped vibratory systems with distinct eigenvalues.

1 INTRODUCTION

Methods for computing the derivatives of natural frequencies and the corresponding mode shapes have been studied in the past 30 years. Finding the derivatives of eigenpairs are essential to determine the sensitivity of dynamic responses of the physical systems. For structural design, these eigenpair derivatives are used to optimize the natural frequencies and the mode shapes of structures by varying design parameters.

Straightforward calculation of eigenvalue derivatives is shown in Fox & Kapoor (1968), Taylor (1972) and Plaut & Huseyin (1973) etc., but the determination of eigenvector derivatives is very complicated because of the singularity problem. Hence there have been suggested a number of different methods to compute the eigenvector derivatives of singular matrix equations.

Rudisill & Chu (1975) presented an algebraic method for the eigenvector derivatives. This method is restricted to the case of non-repeated eigenvalue problem and this method has an asymmetric coefficient matrix. Nelson (1976) solved the same problem; His technique requires only the knowledge of eigenvector to be differentiated and is recommended as an efficient solver for calculating the mode shape derivatives. This method is limited to the distinct eigenvalue problem too. Because of its complicated algorithm, programming code is lengthy and clumsy. Ojalvo (1988) extended Nelson's method to the multiple eigenvalue problem. Mills-Curren (1988) and Dailey (1989) modified Ojalvo's work. Because

those methods are based on Nelson's, their algorithms are so complicated too. In addition to these techniques, modal method (Murthy & Haftka 1988, Lim & Junkins 1987) and its modified one (Wang 1985, Liu *et al.* 1987) approximate the mode shape derivatives by a linear combination of mode shapes. It takes lots of computing time when these approaches are used due to a large number of modes. Additionally, we have an iterative method (Andrew 1978, Tan 1986 and Lee & Jung 1996), whose drawback is its inaccurate solution. Lee & Jung (1997a, b) studied an algebraic method for computing the eigenvalue and eigenvector derivatives of general matrix with non-repeated and repeated eigenvalues. This approach is very efficient and simple and it needs the only corresponding eigenvalue and eigenvector. Furthermore, not only an exact solution is obtained but also numerical stability is proved in their method.

A number of the prescribed methods can be applied to the damped systems; Hallquist (1976) proposed a method for determining the effects of mass modification in viscously damped systems. Recently Zimoch (1987) presented a sensitivity analysis method, which is applied to conservative ones as well as non-conservative systems. It, however, may be restricted to mechanical systems (lumped systems) having only distinct eigenvalues. In other words, implementing this method to the systems with multiple eigenvalues is difficult. Lee & Jung's method (1997a, b) is extended to the damped systems by Lee *et al.* (1999a, b).

In this paper, an improved method over Lee *et al.*

(1999a, b) is studied with reduction of the number of equations for the eigenpair derivatives. The eigenvalue derivatives are obtained apart from the eigenvector derivatives solving two equations in Lee *et al.* (1999a, b). But the eigenpair derivatives are found by solving one modified equation in this paper. Therefore, the FLOPS are reduced for computing to get the eigenpair derivatives while maintaining the advantages of Lee *et al.* (1999a, b). The algebraic equation of the proposed method is efficiently solved by the LDL^T type decomposition method. If the derivatives of stiffness, mass and damping matrices can be obtained analytically, the proposed method can find the exact eigenpair derivatives.

The case of distinct eigenvalue for the damped systems (Lee *et al.* 1999a) is reviewed in the second section of this paper. The proposed method and its stability proof are made in the third section. The results are illustrated with two numerical examples in the next section.

2 PREVIOUS STUDY

As in Fox & Kapoor (1968), Taylor (1972) and Plaut & Huseyin (1973) etc., the calculation of the eigenvalue derivatives is simple. But the calculation of the eigenvector derivatives is complicated because of the singularity of the corresponding equation, therefore many researchers have tried to overcome the problem with various methods.

The finite-difference method (Adelman & Haftka 1986, Lim *et al.* 1987 and Vanhonacker 1980) uses a difference formula to approximate the derivative numerically, which requires calculating the eigenvector at a nominal and at least one perturbed design point. The modal method approximates the mode shape derivatives as a linear combination of mode shapes. The modified modal method was developed to reduce the number of nodes needed to represent the derivative of mode shape. The iterative method approximates the eigenvector derivatives using iteration numerically.

By the research in sensitivity methods, Lee & Jung's method (1997a, b) and their extension to damped systems (Lee *et al.* 1999a, b) is most stable, exact and simple as mentioned in previous section. In this paper, the improved method over Lee *et al.* (1999a) for the damped systems is presented, hence Lee and Jung's method, the base of the proposed method, is reviewed.

2.1 Lee *et al.* (1999a)

Consider a multi-degree-of-freedom damped system described as

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{C}\dot{\mathbf{y}}(t) + \mathbf{K}\mathbf{y}(t) = \mathbf{f}(t), \quad (1)$$

where \mathbf{M} , \mathbf{C} and \mathbf{K} are the matrices of mass, damp-

ing and stiffness respectively, and $n \times n$ symmetric matrices. \mathbf{M} is positive definite and \mathbf{K} is positive definite or semi-positive definite. \mathbf{f} is the excitation vector and \mathbf{y} is the response vector. To determine the eigenvalue and eigenvector sensitivities, first consider the free vibration system. The solution of the free vibration of equation (1) can be assumed as

$$\mathbf{y}(t) = e^{\lambda t} \phi. \quad (2)$$

Substituting equation (2) into equation (1) gives

$$(\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K})\phi = \mathbf{0}, \quad (3)$$

where λ and ϕ are the eigenvalue and eigenvector and both are complex values in general. And in order to determine the eigenvalue derivatives, the differentiating of equation (3) is used. Equation (3) is differentiated with respect to a design parameter p , then

$$\begin{aligned} (\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K}) \frac{\partial \phi_j}{\partial p} = \\ - (2\lambda_j \mathbf{M} + \mathbf{C})\phi_j \frac{\partial \lambda_j}{\partial p} - \left(\lambda_j^2 \frac{\partial \mathbf{M}}{\partial p} + \lambda_j \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p} \right) \phi_j \end{aligned} \quad (4)$$

Premultiplying at each side of equation (4) by ϕ_j^T , the eigenvalue derivative can be obtained as

$$\frac{\partial \lambda_j}{\partial p} = -\phi_j^T \left(\lambda_j^2 \frac{\partial \mathbf{M}}{\partial p} + \lambda_j \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p} \right) \phi_j. \quad (5)$$

But the eigenvector derivative $\partial \phi_j / \partial p$ cannot be found directly from equation (4), since the matrix $\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K}$ is singular. To overcome this problem, the side condition is used in Lee & Jung's method.

To obtain the side condition, one can use state-space-form described by $2n$ -dimensional eigenvalue problem as

$$\begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda \phi \end{Bmatrix} = \lambda \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda \phi \end{Bmatrix}, \quad (6)$$

which can be written conveniently as

$$\mathbf{A}\mathbf{z} = \lambda \mathbf{B}\mathbf{z}, \quad (7)$$

where

$$\mathbf{A} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \mathbf{z} = \begin{Bmatrix} \phi \\ \lambda \phi \end{Bmatrix}. \quad (8)$$

The solutions of the complex eigenvalue problem equation (7) can be found as shown in Caughey & O'Kelly (1960), Wilson & Penzien (1972) and Trail-Nash (1981), and they are distinct and conjugate. The eigenvectors are normalized such as

$$\mathbf{z}_j^T \mathbf{B} \mathbf{z}_j = \begin{Bmatrix} \phi_j \\ \lambda_j \phi_j \end{Bmatrix}^T \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \phi_j \\ \lambda_j \phi_j \end{Bmatrix} = 1, \quad (9)$$

and arranging equation (9) gives

$$\phi_j^T (2\lambda_j \mathbf{M} + \mathbf{C}) \phi_j = 1. \quad (10)$$

Suppose that all eigenpairs and matrices $\partial \mathbf{K} / \partial p$, $\partial \mathbf{M} / \partial p$ and $\partial \mathbf{C} / \partial p$ are known, and all eigenvalues are different, where p is a design parameter. To obtain the derivatives of eigenvector, the differentials

of the normalization condition are used as side condition.

Differentiating equation (10) with respect to design parameter p gives

$$\begin{aligned} & \phi_j^T (2\lambda_j \mathbf{M} + \mathbf{C}) \frac{\partial \phi_j}{\partial p} \\ & + \frac{1}{2} \phi_j^T \left[2 \left(\frac{\partial \lambda_j}{\partial p} \mathbf{M} + \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right) + \frac{\partial \mathbf{C}}{\partial p} \right] \phi_j = 0 \end{aligned} \quad (11)$$

Equation (4) and (11) can be combined as single matrix form as

$$\begin{aligned} & \begin{bmatrix} \lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K} & (2\lambda_j \mathbf{M} + \mathbf{C}) \phi_j \\ \phi_j^T (2\lambda_j \mathbf{M} + \mathbf{C}) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \phi_j}{\partial p} \\ 0 \end{bmatrix} \\ & = \begin{bmatrix} -(2\lambda_j \mathbf{M} + \mathbf{C}) \phi_j \frac{\partial \lambda_j}{\partial p} - \left(\lambda_j^2 \frac{\partial \mathbf{M}}{\partial p} + \lambda_j \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p} \right) \phi_j \\ -\frac{1}{2} \phi_j^T \left[2 \left(\frac{\partial \lambda_j}{\partial p} \mathbf{M} + \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right) + \frac{\partial \mathbf{C}}{\partial p} \right] \phi_j \end{bmatrix} \end{aligned} \quad (12)$$

So the eigenvector derivative $\partial \phi_j / \partial p$ can be obtained directly by solving the algebraic equation. The coefficient matrix of equation (14) is nonsingular.

3 PROPOSED METHOD

3.1 Derivation of the proposed method

Equation (4) is used directly, not premultiplying ϕ_j^T to equation for the derivative of eigenvalue. Rearranging equation (4) gives

$$\begin{aligned} & (\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K}) \frac{\partial \phi_j}{\partial p} + (2\lambda_j \mathbf{M} + \mathbf{C}) \phi_j \frac{\partial \lambda_j}{\partial p} \\ & = - \left(\lambda_j^2 \frac{\partial \mathbf{M}}{\partial p} + \lambda_j \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p} \right) \phi_j \end{aligned} \quad (13)$$

Rearranging equation (11) gives

$$\begin{aligned} & \phi_j^T (2\lambda_j \mathbf{M} + \mathbf{C}) \frac{\partial \phi_j}{\partial p} + \phi_j^T \mathbf{M} \phi_j \frac{\partial \lambda_j}{\partial p} \\ & = -\frac{1}{2} \phi_j^T \left(2\lambda_j \frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p} \right) \phi_j \end{aligned} \quad (14)$$

Because the unknown or interested values are $\partial \phi_j / \partial p$ and $\partial \lambda_j / \partial p$, equation (13) and equation (14) can be combined as single matrix form as follows

$$\begin{aligned} & \begin{bmatrix} \lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K} & (2\lambda_j \mathbf{M} + \mathbf{C}) \phi_j \\ \phi_j^T (2\lambda_j \mathbf{M} + \mathbf{C}) & \phi_j^T \mathbf{M} \phi_j \end{bmatrix} \begin{bmatrix} \frac{\partial \phi_j}{\partial p} \\ \frac{\partial \lambda_j}{\partial p} \end{bmatrix} \\ & = \begin{bmatrix} - \left(\lambda_j^2 \frac{\partial \mathbf{M}}{\partial p} + \lambda_j \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p} \right) \phi_j \\ -\frac{1}{2} \phi_j^T \left(2\lambda_j \frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p} \right) \phi_j \end{bmatrix} \end{aligned} \quad (15)$$

Equation (15) is the key idea of the proposed method. In Lee *et al.*, two equation is needed for the

derivatives of the eigenvalue and eigenvector, because $\partial \lambda_j / \partial p$ is calculated apart from $\partial \phi_j / \partial p$ by pre-multiplying ϕ_j^T at each side of equation (4). But the proposed method finds $\partial \phi_j / \partial p$ and $\partial \lambda_j / \partial p$ at once by solving the single matrix, one algebraic equation, composed of the differentials of the eigenvalue problem and normalization condition.

The proposed method has the characteristics of not only finding exact solutions, having the numerical stability and having a symmetric coefficient matrix which are those of Lee *et al.*, but also being more efficient for the time required to calculate the eigenpair derivatives because of being composed of more simple equation.

The numerical stability of the proposed method is proved in next section.

3.2 Numerical stability of the proposed method

To prove that the coefficient matrix $\mathbf{A}^{\#}$ of equation (15) is nonsingular, introduce the nonsingular square matrix \mathbf{Y} , $\det(\mathbf{Y}) \neq 0$. Matrix \mathbf{Y} is used in the determinant property,

$$\det(\mathbf{Y}^T \mathbf{A}^{\#} \mathbf{Y}) = \det(\mathbf{Y}^T) \det(\mathbf{A}^{\#}) \det(\mathbf{Y}).$$

If $\det(\mathbf{Y}^T \mathbf{A}^{\#} \mathbf{Y}) \neq 0$, the determinant of $\mathbf{A}^{\#}$ is nonzero because the determinant of \mathbf{Y} is nonzero.

Nonsingular matrix \mathbf{Y} is assumed as

$$\mathbf{Y} = \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \quad (16)$$

where $\Psi = [\psi_1 \ \psi_2 \ \dots \ \psi_{n-1} \ \phi_j]$, ϕ_j is the j th eigenvector of the system and ψ 's are arbitrary vectors to be independent of ϕ_j . Ψ is a $n \times n$ matrix, \mathbf{Y} is a $(n+1) \times (n+1)$ matrix.

Pre- and post-multiplying \mathbf{Y}^T and \mathbf{Y} to $\mathbf{A}^{\#}$ yields

$$\begin{aligned} \mathbf{Y}^T \mathbf{A}^{\#} \mathbf{Y} &= \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^T \left[\begin{array}{c|c} \lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K} & (2\lambda_j \mathbf{M} + \mathbf{C}) \phi_j \\ \hline \phi_j^T (2\lambda_j \mathbf{M} + \mathbf{C}) & \phi_j^T \mathbf{M} \phi_j \end{array} \right] \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \\ &= \left[\begin{array}{c|c} \Psi^T (\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K}) \Psi & \Psi^T (2\lambda_j \mathbf{M} + \mathbf{C}) \phi_j \\ \hline \phi_j^T (2\lambda_j \mathbf{M} + \mathbf{C}) \Psi & \phi_j^T \mathbf{M} \phi_j \end{array} \right] \end{aligned} \quad (17)$$

The last column and row of the matrix $\Psi^T (\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K}) \Psi$ is zero because of ϕ_j which is the last column of Ψ . That is

$$\Psi^T (\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K}) \Psi = \begin{bmatrix} \tilde{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \quad (18)$$

where $\tilde{\mathbf{A}}$ is a nonzero $(n-1) \times (n-1)$ submatrix. And λ_j is a distinct eigenvalue of the system, therefore the matrices $\Psi^T (\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K}) \Psi$ of order n have a rank $n-1$, they are singular. But the submatrix $\tilde{\mathbf{A}}$ of order $n-1$ has full rank $n-1$, and it is nonsingular, $\det(\tilde{\mathbf{A}}) \neq 0$.

By normalization condition the last elements of the column vector $\Psi^T (2\lambda_j \mathbf{M} + \mathbf{C}) \phi_j$ and the row vector $\phi_j^T (2\lambda_j \mathbf{M} + \mathbf{C}) \Psi$ are unity.

Table 1. The lowest ten eigenvalue and their derivatives

Mode Number	Eigenvalue	Eigenvalue derivative (Lee&Jung's method)	Eigenvalue derivative (Proposed method)
1	-0.0004 - 2.6248i	-0.0138 -52.4963i	-0.0138 -52.4963i
2	-0.0004 + 2.6248i	-0.0138 +52.4963i	-0.0138 +52.4963i
3	-0.0136 -16.4491i	-5.4111e-1 -3.2896e+2i	-5.4111e-1 -3.2896e+2i
4	-0.0136 +16.4491i	-5.4111e-1 +3.2896e+2i	-5.4111e-1 +3.2896e+2i
5	-0.0345 -26.2358i	4.7702e-7 +2.9684e-8i	4.7702e-7 +2.9684e-8i
6	-0.0345 +26.2358i	4.7212e-7 +1.5490e-7i	4.7212e-7 +1.5490e-7i
7	-0.1061 -46.0558i	-4.2416e+0 -9.2096e+2i	-4.2416e+0 -9.2096e+2i
8	-0.1061 +46.0558i	-4.2416e+0 +9.2096e+2i	-4.2416e+0 +9.2096e+2i
9	-0.4073 -90.2444i	-1.6284e+1 -1.8043e+3i	-1.6284e+1 -1.8043e+3i
10	-0.4073 +90.2444i	-1.6284e+1 +1.8043e+3i	-1.6284e+1 +1.8043e+3i

$$\Psi^T (2\lambda_j \mathbf{M} + \mathbf{C}) \phi_j = \begin{Bmatrix} \tilde{\mathbf{b}} \\ 1 \end{Bmatrix}, \quad (19)$$

$$\phi_j^T (2\lambda_j \mathbf{M} + \mathbf{C}) \Psi = \begin{Bmatrix} \tilde{\mathbf{b}}^T \\ 1 \end{Bmatrix}^T \quad (20)$$

where $\tilde{\mathbf{b}}$ is nonzero vector. Substituting equation (18), (19) and (20) into equation (17) gives

$$\mathbf{Y}^T \mathbf{A}^* \mathbf{Y} = \begin{bmatrix} \tilde{\mathbf{A}} & \mathbf{0} & \tilde{\mathbf{b}} \\ \mathbf{0} & \mathbf{0} & 1 \\ \tilde{\mathbf{b}}^T & 1 & \phi_j^T \mathbf{M} \phi_j \end{bmatrix}. \quad (21)$$

Applying the determinant property of partitioned matrix gives that

$$\det(\mathbf{Y}^T \mathbf{A}^* \mathbf{Y}) =$$

$$\det \begin{bmatrix} 0 & 1 \\ 1 & \phi_j^T \mathbf{M} \phi_j \end{bmatrix} \det \left(\tilde{\mathbf{A}} - \begin{bmatrix} \mathbf{0} & \tilde{\mathbf{b}} \\ 1 & \phi_j^T \mathbf{M} \phi_j \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{b}}^T \end{bmatrix} \right) \quad (22)$$

where

$$\begin{bmatrix} \mathbf{0} & \tilde{\mathbf{b}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & \phi_j^T \mathbf{M} \phi_j \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{b}}^T \end{bmatrix} = 0 \quad (23)$$

$$\det \begin{bmatrix} 0 & 1 \\ 1 & \phi_j^T \mathbf{M} \phi_j \end{bmatrix} = -1. \quad (24)$$

Table 2. Some components of the first eigenvector and its derivatives

Eqn. number	Eigenvector	Eigenvector derivative (Lee&Jung's method)	Eigenvector derivative (Proposed method)
1	0	0	0
2	0	0	0
3	1.5133e-05+1.5133e-05i	-3.0267e-04-3.0267e-04i	-3.0267e-04-3.0267e-04i
4	1.2036e-04+1.2036e-04i	-0.0024-0.0024i	-0.0024-0.0024i
5	0	0	0
⋮			
157	0	0	0
158	0	0	0
159	0.0139 + 0.0139i	-0.2787 - 0.2787i	-0.2787 - 0.2787i
160	0.0019 + 0.0019i	-0.0384 - 0.0384i	-0.0384 - 0.0384i

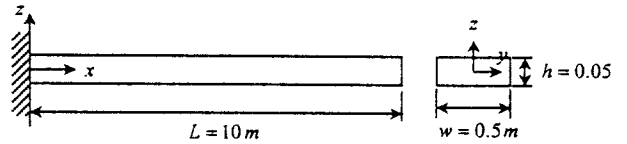


Figure 1. Cantilever beam with the thickness h as the design parameter.

Therefore rearranging equation (22) gives

$$\det(\mathbf{Y}^T \mathbf{A}^* \mathbf{Y}) = -\det(\tilde{\mathbf{A}}) \neq 0. \quad (25)$$

The determinant of \mathbf{A}^* thus is not equal to zero, in other words, the matrix \mathbf{A}^* is nonsingular.

4 NUMERICAL EXAMPLE

The efficiency and exactness of Lee *et al.* are verified in numerical examples of Lee *et al.* (1999a, b). In this section, the results by those method with cantilever beam are presented, for the case of the distinct natural frequencies. Pentium 120 having CPU capacity 120Mhz with RAM 40Mega is used for computation.

A cantilever beam with 40 elements is considered as shown in Figure 2. The number of nodes is 41 and each node has four degrees of freedom (y-translation, z-rotation, z-translation, y-rotation); the total number of degrees of freedom is 160(one node

is fixed). For this example, Young's modulus ($2.1 \times 10^{11} \text{ N/m}^2$), the mass density ($7.85 \times 10^3 \text{ kg/m}^3$) and the Poisson's ratio 0.3 are used. The length of the beam is 10m , width 0.5m and depth 0.05m .

Rayleigh damping can be assumed as the damping matrix, which is of the form as

$$C = \alpha K + \beta M, \quad (26)$$

where α and β are the Rayleigh coefficients and $\alpha = \beta = 0.01$. The design parameter is the depth of beam, h .

Some sensitivity results are represented in Table 1 and Table 2.

As shown in Table 1 and Table 2, we can see that the results by Lee *et al.* and the proposed method are the same. It is exact solution. In Table 2, the first, second and fifth components and so on, are zero because first mode vibrate in z-direction (z-translation and y-rotation). The first and second component are those of y-translation and z-rotation respectively. The analysis time is 223.33 seconds in

Lee *et al.* and 164.89 seconds in the proposed method for 160 eigenpair derivatives. The analysis time of the proposed method is less than that of Lee *et al.* by 26.17%.

CPU time required to calculate the 160 eigenpair derivatives is summarized for each operation in Table 3. The result of Table 3 is the average of those of ten times analysis, and standard deviations are 1.56 and 1.29 seconds for total CPU time of Lee *et al.* and the proposed method respectively.

Table 3. CPU time spent on the calculation of the first 160 eigenpair derivatives

Method	Operations	CPU time (sec)
Lee & Jung's Method	$\frac{\partial \lambda_j}{\partial p} = -\phi_j^T \left(\lambda_j^2 \frac{\partial M}{\partial p} + \lambda_j \frac{\partial C}{\partial p} + \frac{\partial K}{\partial p} \right) \phi_j$	33.89
	$A^* = \begin{bmatrix} \lambda_j^2 M + \lambda_j C + K & (2\lambda_j M + C) \phi_j \\ \phi_j^T (2\lambda_j M + C) & 0 \end{bmatrix}$	61.01
	$f_j = \begin{cases} -(2\lambda_j M + C) \phi_j \frac{\partial \lambda_j}{\partial p} - \left(\lambda_j^2 \frac{\partial M}{\partial p} + \lambda_j \frac{\partial C}{\partial p} + \frac{\partial K}{\partial p} \right) \phi_j \\ -\frac{1}{2} \phi_j^T \left[2 \left(\frac{\partial \lambda_j}{\partial p} M + \lambda_j \frac{\partial M}{\partial p} \right) + \frac{\partial C}{\partial p} \right] \phi_j \end{cases}$	47.09
	$\begin{Bmatrix} \frac{\partial \phi_j}{\partial p} \\ 0 \end{Bmatrix} = [A^*]^{-1} f_j$	81.34
	Total	223.33
Proposed Method	$A^* = \begin{bmatrix} \lambda_j^2 M + \lambda_j C + K & (2\lambda_j M + C) \phi_j \\ \phi_j^T (2\lambda_j M + C) & \phi_j^T M \phi_j \end{bmatrix}$	53.62
	$\bar{f}_j = \begin{cases} - \left(\lambda_j^2 \frac{\partial M}{\partial p} + \lambda_j \frac{\partial C}{\partial p} + \frac{\partial K}{\partial p} \right) \phi_j \\ -\frac{1}{2} \phi_j^T \left(2\lambda_j \frac{\partial M}{\partial p} + \frac{\partial C}{\partial p} \right) \phi_j \end{cases}$	40.60
	$\begin{Bmatrix} \frac{\partial \phi_j}{\partial p} \\ \frac{\partial \lambda_j}{\partial p} \end{Bmatrix} = [A^*]^{-1} \bar{f}_j$	70.67
	Total	164.89

5 CONCLUSION

To calculate the derivatives of eigenvalue and eigenvector, two equations were used in the previous work. But, by unifying the two equations by one, a simple algorithm is developed and simulated for the systems with non-repeated eigenvalues.

The proposed method is an improvement of Lee *et al.*: An exact solution is obtained and the numerical stability is proved as in Lee *et al.* with an simplified algorithm. Additionally, CPU time to compute the eigenpair derivatives is remarkably reduced due to the simplification of algorithm in the proposed method as compared with Lee *et al.*

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