The Fourteenth KKNN Symposium on Civil Engineering
November 5-7, 2001, Kyoto, Japan

ALGEBRAIC METHOD FOR SENSITIVITY ANALYSIS OF EIGENSYSTEMS WITH REPEATED EIGENVALUES

Kang-Min Choi, Woon-Hak Kim and In-Won Lee
Department of Civil Engineering, Korea Advanced Institute of Science and Technology, 373-1 Guseong-dong, Yuseong-gu, Daejeon 305-701, Korea
kmchoi@mail.kaist.ac.kr, whkim@hnu.hankyong.ac.kr, iwlee@kaist.ac.kr

Abstract

A simplified method is presented for the computation of eigenvalue and eigenvector derivatives associated with repeated eigenvalues. In the proposed method, adjacent eigenvectors and orthonormal conditions are used to compose an algebraic equation whose order is (n+m)×(n+m), where n is the number of coordinates and m is the number of multiplicity of the repeated eigenvalue. One algebraic equation developed can be computed eigenvalue and eigenvector derivatives simultaneously. Since the coefficient matrix of the proposed equation is symmetric and based on N-space, this method is very efficient compared to previous methods. Moreover the numerical stability of the method is guaranteed because the coefficient matrix of the proposed equation is non-singular. To verify the effectiveness of the proposed method, the finite element model of the cantilever beam is considered as a numerical example. The design parameter of the cantilever beam is its width.

Introduction

The sensitivity of eigenvalue problem with repeated eigenvalues has been a focus of recent interest. The most common circumstances under which repeated eigenvalues or nearly equal eigenvalues occur in typical structural or mechanical systems are instances where system symmetry exists, such as structures with two or more planes of reflective or cyclic symmetric or in the limiting case of axisymmetric bodies or certain reasons. In this case, since the eigenspace spanned by eigenvectors corresponding to repeated eigenvalues is degenerate, any linear combination of eigenvectors can be an eigenvector. For the eigenvector derivative to be found, the adjacent eigenvectors which lie "adjacent" to the m(multiplicity of repeated eigenvalues) distinct eigenvectors appearing when a design parameter varies must be calculated first. To do so, the approximate eigenvectors could be varied continuously by varying the design parameter.

For the real symmetric case, a generalization of Nelson's method was obtained by Ojalvo and amended by Mills-Curren and Dailey. These methods are lengthy and complicated for finding eigenvector derivatives and clumsy for programming, because they basically follow Nelson's algorithm. Lee *et al.* developed analytical method that give exact solutions while it maintains N-space, but it finds eigenvalue derivative from classical method as before.

In this paper, an efficient algebraic method for the eigenpair sensitivities of damped systems with repeated eigenvalues is presented. Contrary to previous methods, the proposed method finds the eigenvalue and eigenvector sensitivities simultaneously from one equation. The proposed method doesn't use a state space equation(2N-space), instead of it, the method maintain N-space because a singularity problem is solved by using only one side condition. If the derivatives of the stiffness, mass and damping matrices can be analytically found, the proposed method can find the exact eigenpair derivatives. And it only requires the corresponding eigenpair information differently from modal methods.

The second section of this paper presents the proposed sensitivity analysis method of damped systems with repeated eigenvalues. In the next section, the numerical stability of the proposed method is presented, and finally numerical examples.

Eigenpair sensitivity in damped systems with repeated eigenvalues

When an eigenvalue has multiplicity *m* and a design parameter is perturbed, the corresponding eigenvectors may split into as many as *m* distinct eigenvectors. For derivatives of the eigenvectors to be responsible, the eigenvectors must be laid adjacent to the *m* distinct eigenvectors that appear when a design parameter varies. Otherwise, the eigenvectors would jump discontinuously with a varying design parameter. Here the derivatives of these adjacent eigenvectors are sought.
The eigenvalue problem of a damped system can be expressed as

$$(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})\phi = 0 \tag{1}$$

where M, C and K are the matrices of mass, damping and stiffness, respectively, and these are (n×n) symmetric matrices. M is positive definite and K is positive definite or semi-positive definite. The first step in finding derivatives of eigenvectors of repeated eigenvalues is to find corresponding adjacent eigenvectors. Suppose that all eigenpairs are known and multiplicity of the eigenvalue λ_m is *m*. Define the following eigenvalue problem where Φ_m is the matrix of eigenvectors corresponding to the repeated eigenvalues, hence, its order (n×m)

$$\mathbf{M}\Phi_m\Lambda_m^2 + \mathbf{C}\Phi_m\Lambda_m + \mathbf{K}\Phi_m = 0 \tag{2}$$

where

$$\Lambda_m = \lambda_m\mathbf{I}_m \quad \text{and} \quad \Phi_m = [\phi_{i+1}\ \phi_{i+2}\ \cdots\ \phi_{i+m}] \tag{3}$$

$\mathbf{I}_m$ is the identity matrix of order m and λ_m is the eigenvalue of multiplicity *m* for the eigenspace spanned by the columns of Φ_m. The orthonormal condition for the matrix Φ_m is as follows:

$$\Phi_m^T(2\lambda_m\mathbf{M} + \mathbf{C})\Phi_m = \mathbf{I}_m \tag{4}$$

Adjacent eigenvectors can be expressed in terms of Φ_m by an orthogonal transformation such as

$$\mathbf{X} = \Phi_m\mathbf{T} \tag{5}$$

where T is an orthonormal transformation matrix and its order *m*;
The columns of X are the adjacent eigenvectors for which a derivative can be defined.

It is natural that the adjacent eigenvectors also satisfy the orthonormal condition:

$$\mathbf{X}^T(2\lambda_m\mathbf{M}+\mathbf{C})\mathbf{X}=\mathbf{T}^T\Phi_m^T(2\lambda_m\mathbf{M}+\mathbf{C})\Phi_m\mathbf{T}=\mathbf{T}^T\mathbf{T}=\mathbf{I}_m \tag{6}$$

The next procedure is to find T and then to find X and $\Lambda_{m,\alpha}$. If design parameter α varies, $\Lambda_{m,\alpha}$ is expressed as

$$\Lambda_{m,\alpha}=\mathbf{diag}\left(\lambda_{1+1,\alpha},\ \lambda_{1+2,\alpha},\ \cdots,\ \lambda_{1+m,\alpha}\right) \tag{7}$$

where, $(\bullet)_{,\alpha}$ represents the derivative of $(\bullet)$ with respect design parameter α. Consider another eigenvalue problem to find X and $\Lambda_{m,\alpha}$.

$$\mathbf{MX}\Lambda_m^2+\mathbf{CX}\Lambda_m+\mathbf{KX}=0 \tag{8}$$

where the order of adjacent eigenvector matrix X is (n×m) and the order of eigenvalue matrix Λ_m is (m×m). Differentiating above eigenvalue problem with respect to the design parameter α, and rearranging yields

$$(\lambda_m^2\mathbf{M}+\lambda_m\mathbf{C}+\mathbf{K})\mathbf{X}_{,\alpha}+(2\lambda_m\mathbf{M}+\mathbf{C})\mathbf{X}\Lambda_{m,\alpha}=-\left(\lambda_m^2\mathbf{M}_{,\alpha}+\lambda_m\mathbf{C}_{,\alpha}+\mathbf{K}_{,\alpha}\right)\mathbf{X} \tag{9}$$

Pre-multiplying at each side of equation (9) by Φ_m^T and substituting $\mathbf{X}=\Phi_m\mathbf{T}$ into it gives a new eigenvalue problem such as

$$\mathbf{DT}=\mathbf{ET}\Lambda_{m,\alpha} \tag{10}$$

where

$$\mathbf{D}=\Phi_m^T\left(\lambda_m^2\mathbf{M}_{,\alpha}+\lambda_m\mathbf{C}_{,\alpha}+\mathbf{K}_{,\alpha}\right)\Phi_m \quad \text{and} \quad \mathbf{E}=-\Phi_m^T(2\lambda_m\mathbf{M}+\mathbf{C})\Phi_m=-\mathbf{I}_m \tag{11}$$

The orthogonal transformation matrix T can be obtained by solving equation (10), and then the adjacent eigenvectors by relation $\mathbf{X}=\Phi_m\mathbf{T}$.
The proposed method starts with the equations of the derivative of the eigenvalue problem composed of the system matrices and the adjacent eigenvectors, equation (9), and the orthonormal condition, equation (6). Differenting equation (6) with respect to the design parameter gives

$$\mathbf{X}^T(2\lambda_m\mathbf{M}+\mathbf{C})\mathbf{X}_{,\alpha}+\mathbf{X}^T\mathbf{MX}\Lambda_{m,\alpha}=-0.5\mathbf{X}^T\left(2\lambda_m\mathbf{M}_{,\alpha}+\mathbf{C}_{,\alpha}\right)\mathbf{X} \tag{12}$$

Since the unknown or interested values are $\mathbf{X}_{,\alpha}$ and $\Lambda_{m,\alpha}$, equation (9) and equation (12) can be combined into a single matrix form as follows:

$$\begin{bmatrix}\lambda_m^2\mathbf{M}+\lambda_m\mathbf{C}+\mathbf{K} & (2\lambda_m\mathbf{M}+\mathbf{C})\mathbf{X}\\ \mathbf{X}^T(2\lambda_m\mathbf{M}+\mathbf{C}) & \mathbf{X}^T\mathbf{MX}\end{bmatrix}\begin{Bmatrix}\mathbf{X}_{,\alpha}\\ \Lambda_{m,\alpha}\end{Bmatrix}=\begin{bmatrix}-\left(\lambda_m^2\mathbf{M}_{,\alpha}+\lambda_m\mathbf{C}_{,\alpha}+\mathbf{K}_{,\alpha}\right)\mathbf{X}\\ -0.5\mathbf{X}^T\left(2\lambda_m\mathbf{M}_{,\alpha}+\mathbf{C}_{,\alpha}\right)\mathbf{X}\end{bmatrix} \tag{13}$$

where the order of coefficient matrix on the left side of equation (13) is (n+m)×(n+m) and the matrix of the right side of equation is (n+m)×m. The derivatives $\mathbf{X}_{,\alpha}$ and $\Lambda_{m,\alpha}$ can be found by solving equation (13).

Contrary to previous method, the sensitivities of the eigenvalue and eigenvector can be obtained simultaneously from one augmented equation. It maintains N-space without use of state space equation and finds the eigenpair derivatives simultaneously.

The proposed method requires only corresponding eigenpair information differently from modal methods and gives exact solution and guarantees numerical stability, which is proved in the next section.

Numerical Stability of the proposed method

Numerical stability is guaranteed by proving non-singularity of the coefficient matrix A^* in equation (13). To prove that the coefficient matrix A^* is non-singular, introduce the determinant property such as

$$\det(\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}) = \det(\mathbf{Y}^T)\det(\mathbf{A}^*)\det(\mathbf{Y}) \tag{14}$$

If $\det(\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}) \neq 0$ is proved with an arbitrary non-singular matrix Y, $\det(\mathbf{A}^*) \neq 0$ is proved. In this paper, the arbitrary non-singular matrix Y is assumed as

$$\mathbf{Y} = \begin{bmatrix} \Psi & 0 \\ 0 & \mathbf{I}_m \end{bmatrix} \tag{15}$$

where $\mathbf{I}_m$ is an identity matrix of order m and Ψ is a set of arbitrary independent vectors containing the adjacent eigenvectors of repeated eigenvalue λ_m of the systems, as follows

$$\Psi = [\phi_1 \ \phi_2 \ \cdots \ \phi_{n-m} \ \mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_m] \text{ when } \mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_m] \tag{16}$$

where ψ's are arbitrary independent vectors chosen to be independent to the adjacent eigenvector x's. Since all the columns of the matrix Y are independent vectors, matrix Y is non-singular and it is invertible. Pre- and post-multiplying Y^T and Y to A^* yields

$$\begin{aligned} \mathbf{Y}^T\mathbf{A}^*\mathbf{Y} &= \begin{bmatrix} \Psi & 0 \\ 0 & \mathbf{I}_m \end{bmatrix}^T \left[\begin{array}{c|c} \lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K} & (2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X} \\ \hline \mathbf{X}^T(2\lambda_m\mathbf{M} + \mathbf{C}) & \mathbf{X}^T\mathbf{M}\mathbf{X} \end{array}\right] \begin{bmatrix} \Psi & 0 \\ 0 & \mathbf{I}_m \end{bmatrix} \\ &= \left[\begin{array}{c|c} \Psi^T(\lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K})\Psi & \Psi^T(2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X} \\ \hline \mathbf{X}(2\lambda_m\mathbf{M} + \mathbf{C})\Psi & \mathbf{X}^T\mathbf{M}\mathbf{X} \end{array}\right] \end{aligned} \tag{17}$$

It is obvious that the last m columns and rows of the matrix $\Psi^T(\lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K})\Psi$ all have zero elements, which are provided by the eigenvalue problem $(\lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K})\mathbf{X} = 0$, as follows

$$\Psi^T(\lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K})\Psi = \begin{bmatrix} \tilde{\mathbf{A}} & 0 \\ 0 & 0 \end{bmatrix} \tag{18}$$

where $\tilde{\mathbf{A}}$ is a non-zero (n-m)x(n-m) submatrix. The submatrix $\tilde{\mathbf{A}}$ is a non-singular matrix having order of (n-m) and rank of (n-m), since it is given by eliminating the columns and rows having all zero elements from $\Psi^T(\lambda_m^2\mathbf{M} + \lambda_m\mathbf{C} + \mathbf{K})\Psi$ of order n and rank (n-m). That is, $\det(\tilde{\mathbf{A}}) \neq 0$.

By the normalization condition,

$$\Psi^T(2\lambda_m\mathbf{M} + \mathbf{C})\mathbf{X} = \begin{bmatrix} \tilde{\mathbf{B}} \\ \mathbf{I}_m \end{bmatrix} \text{ and } \mathbf{X}^T(2\lambda_m\mathbf{M} + \mathbf{C})\Psi = \begin{bmatrix} \tilde{\mathbf{B}} \\ \mathbf{I}_m \end{bmatrix}^T \tag{19}$$

where $\tilde{\mathbf{B}}$ is generally a non-zero rectangular matrix. Substituting equation (18) and

(19), into equation (17) yields

$$\mathbf{Y}^{\mathrm{T}}\mathbf{A}^{*}\mathbf{Y} = \left[\begin{array}{c|cc} \widetilde{\mathbf{A}} & 0 & \widetilde{\mathbf{B}} \\ \hline 0 & 0 & \mathbf{I}_m \\ \widetilde{\mathbf{B}}^{\mathrm{T}} & \mathbf{I}_m & \mathbf{X}^{\mathrm{T}}\mathbf{M}\mathbf{X} \end{array}\right] \tag{20}$$

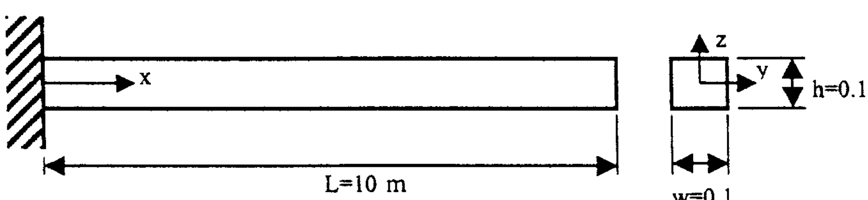
To find the determinant of the matrix, apply the determinant property of partitioned matrices. Hence the determinant of equation (20) can be rewritten as

$$\begin{aligned} \det(\mathbf{Y}^{\mathrm{T}}\mathbf{A}^{*}\mathbf{Y}) &= \det(\widetilde{\mathbf{A}}) \times \det\left(\begin{bmatrix} 0 & \mathbf{I}_m \\ \mathbf{I}_m & \mathbf{X}^{\mathrm{T}}\mathbf{M}\mathbf{X} \end{bmatrix} - \begin{bmatrix} 0 \\ \widetilde{\mathbf{B}}^{\mathrm{T}} \end{bmatrix} \left[\widetilde{\mathbf{A}}\right]^{-1} \begin{bmatrix} 0 & \widetilde{\mathbf{B}} \end{bmatrix} \right) \\ &= -\det(\widetilde{\mathbf{A}}) \neq 0 \end{aligned} \tag{21}$$

The determinant of A^* thus is not equal to zero. In other words, the matrix A^* is non-singular. The proof is completed mathematically for the numerical stability of the proposed algorithm in the case of repeated eigenvalues.

Numerical Examples

To verify the effectiveness of the proposed method, the finite element model of a cantilever beam as the proportionally damped system is presented. It is FEM model composed of 20 elements and 21 nodes. Each node has four degrees of freedom (y-translation, z-translation, y-rotation and z-rotation). The structure has 80 degrees of freedom. Rayleigh damping($C=\alpha K+\beta M$) is considered. The design parameter is the beam width w.



System Data
Number of nodes: 21
Number of elements: 20
Number of DOF: 80

Material Properties
Young's Modulus: $E=2.10\times10^{11}$
Mass density: $\rho=7.85\times10^{3}$ kg/m^3
Design parameter: width of beam

Figure 1. Cantilever beam with width w as the design parameter.

Some results are shown in Table 1. The lowest 12 eigenvalues and their derivatives of the cantilever beam are listed in the second and third columns of Table 1. One can see that the derivatives of the repeated eigenvalues are different in that one is close to zero while the other is not. Since the design parameter is the width of the beam w, when w is varied, the repeated eigenvalues are split into distinct ones as the cross-section of the beam is no longer square after changing the width. To illustrate the sensitivity analysis results, the actual and approximate values of the changed system of $\Delta w/w=0.01$ are represented in the fourth and fifth columns of Table 1. The next two columns are the errors of the approximations. The errors are reasonably small and one can say that the proposed method gives good results for the case of repeated eigenvalues.

Table 1. The lowest 12 eigenvalues of the initial and changed cantilever beam and results of the sensitivity analysis

Mode number	Initial System		Changed System		Error of Approximation	
	Eigenvalue	Eigenvalue Derivative	Eigenvalue	Approximated Eigenvalue	Eigenvalue	Eigenvector
1,2	-1.4279e-03 ∓j5.2496e-00	-2.8057e-10 ±j3.5347e-10	-1.4279e-03 ∓j5.2496e-00	-1.4279e-03 ∓j5.2496e-00	2.2283e-11	3.7376e-05
3,4	-1.4279e-03 ∓j5.2496e-00	-2.2756e-02 ∓j5.2494e+01	-1.4556e-03 ∓j5.3021e-00	-1.4555e-03 ∓j5.3021e-00	2.6622e-08	1.0000e-04
5,6	-5.4154e-02 ∓j3.2895e+01	-6.6265e-10 ∓j2.3445e-10	-5.4154e-02 ∓j3.2895e+01	-5.4154e-02 ∓j3.2895e+01	3.6899e-12	3.7376e-05
7,8	-5.4154e-02 ∓j3.2895e+01	-1.0818e+00 ∓j3.2886e+02	-5.5241e-02 ∓j3.3224e+01	-5.5236e-02 ∓j3.3224e+01	1.6763e-07	1.0001e-04
9,10	-4.2409e-01 ∓j9.2090e+01	6.9247e-10 ±j6.9600e-10	-4.2409e-01 ∓j9.2090e+01	-4.2409e-01 ∓j9.2090e+01	9.1432e-12	3.7376e-05
11,12	-4.2409e-01 ∓j9.2090e+01	-8.4753e+00 ∓j9.2029e+02	-4.3261e-01 ∓j9.3010e+01	-4.3256e-01 ∓j9.3010e+01	4.6508e-07	1.0002e-04

Conclusion

This paper proposes a simple algorithm for the calculation of eigenpair derivatives of the damped system with repeated eigenvalues. The proposed method finds exact eigenpair derivatives of the system by solving the linear algebraic equation without any numerical instability. In addition, derivatives of eigenvalues and eigenvectors can be obtained simultaneously from one augmented equation. This approach avoids the use of state space equation and considers the damping problem explicitly by introducing a side condition of differentiation of normalization condition. Thus computation for the equation with N-order can be maintained and the computer storage and analysis time required of the proposed method are smaller than those of previous methods. The proposed method can be inserted easily into a commercial FEM code since it finds the exact solution and treats a symmetric matrix.

Acknowledgements

This research was supported by the National Research Laboratory for Aseismic Control of Structures. The support is deeply appreciated.

References

R. L. Dailey (1989), "Eigenvector derivatives with repeated eigenvalues," *AIAA Journal*, **27**, 486-491.

M. S. Jankovic (1994), "Exact *n*th Derivatives of Eigenvalues and Eigenvectors," Journal of Guidance, Control, and Dynamics, **17**, 136-144.

I. W. Lee, D. O. Kim and G. H. Jung (1999), "Natural Frequency and Mode Shape Sensitivities of Damped Systems: Part II, Multiple Natural Frequencies," *Journal of Sound and Vibration*, **223**, 413-424.