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MODIFIED STURM SEQUENCE PROPERTY FOR DAMPED SYSTEMS

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ABSTRACT

Most of the eigenvalue analysis methods for the undamped or proportionally damped systems use the well-known Sturm sequence property to check the missed eigenvalues when only a set of the lowest modes is to be used for large structures. However, in the case of the non-proportionally damped systems such as the soil-structure interaction system, the structural control system and the composite structures, no counterpart of the Sturm sequence property for undamped systems has been developed yet. Hence, when some important modes are missed for those systems, it may leads to poor results in dynamic analysis. In this paper, a technique for calculating the number of eigenvalues inside the open disk of arbitrary radius for the eigenproblem with the damping matrix is proposed by applying Chen's algorithm and Gleyse's theorem. To verify the applicability of the proposed method, two numerical examples are considered.

KEYWORDS

Sturm sequence property, non-proportional damping, eigenproblem with damping, characteristic polynomial, missed eigenvalues, Schur-Cohn matrix.

INTRODUCTION

To obtain the dynamic response of a large civil structure, it is economic and efficient to superpose the results of a few lowest modes. Therefore, there has been proposed many eigensolution techniques which can find only a set of the lowest modes. The Lanczos and subspace method are belong to this type of technique. In these techniques, however, some important modes can be missed in the calculation process, which may lead to poor results in dynamic analysis. Hence, a checking technique for missed eigenvalues is required in finding the missed one. For the case of the undamped or proportionally damped systems, it can be easily found by using the Sturm sequence property. However, in the case of the non-proportionally damped systems such as the soil-structure interaction system, the structural control system and composite structures, no counterpart of the Sturm sequence property for undamped systems has been developed yet. Hence, when some important modes are missed for those systems, it may leads to poor results in dynamic analysis. A number of researchers have been performed to solve the eigenproblem with the damping matrix, whereas there have been few studies on a technique to calculate the number of eigenvalues in this case in the literature. Tsai and Chen proposed the extended Sturm sequence property that can determine the root distribution of a polynomial on some specified lines of the complex plane. However, this extended property can not be applied to the nonproportionally damped system because it is very difficult to find the specified line of the complex plane in this case and the Sturm sequence cannot be formed by factorizing the considered matrix in the field of the complex arithmetic computation. Jung et al. proposed a technique of checking missed eigenvalues for eigenproblem with damping matrix using argument principle. This method requires a selection of checking points and the LDL^T factorization of the characteristic polynomial at those points. The accuracy of the method increases with the number of checking points, so it need more factorization processes to get more exact results. In this paper, Gleyse's theorem , which can count the number of zeros of a characteristic polynomial inside an open unit disk, is used to calculate the number of eigenvalues for eigenproblem with the damping matrix. The characteristic polynomial of an eigenvalue problem is determined by using Chen's algorithm which is considered as both stable and effective. The determinants(minors) of the leading principal submatrices of order i in the Schur-Cohn matrix can be easily calculated by the LDL^T factorization process and the final result obtained is very similar to the Sturm sequence property for the undamped systems. This paper is organized as follows. The modified Sturm sequence property is presented and discussed. A numerical examples are analyzed to verify the effectiveness of the proposed method. Finally, the concluding remarks are expressed.

MODIFIED STURM SEQUENCE PROPERTY FOR DAMPED SYSTEMS

The equations of motion of damped systems

In the analysis of dynamic response of structural system, the equation of motion of damped systems can be written as:

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = 0, \tag{1}$$

where M , K and C are the ($n \times n$) mass, stiffness and nonclassical damping matrices, respectively, and $\ddot{u}(t)$, $\dot{u}(t)$ and $u(t)$ are the ($n \times 1$) acceleration, velocity and displacement vectors, respectively. To find the solution of the free vibration of the system, we consider the following quadratic eigenproblem:

$$\lambda^2 M\phi + \lambda C\phi + K\phi = 0 \tag{2}$$

in which λ and ϕ are the eigenvalue and eigenvector of the system. There are $2n$ eigenvalues for the system with n degrees of freedom and these occur either in real pairs or in complex conjugate pairs, depending upon whether they correspond to overdamped or undamped modes. The common practice is to reformulate the quadratic system of equation to a linear one by doubling the order of the system such as:

$$\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix} = \lambda \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix} \tag{3}$$

In general, M and C are nonsingular, that is $\det(M) \neq 0$ and $\det(C) \neq 0$, so the above equation can be changed to the form of a standard eigenproblem:

$$\begin{bmatrix} C & M \\ M & 0 \end{bmatrix}^{-1} \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix} = \lambda \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix}. \tag{4}$$

After some operations, the resulting standard eigenvalue problem is :

$$A\psi = \lambda\psi, \tag{5}$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \psi = \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix}. \tag{6}$$

Observing the above Equation (6), when the mass matrix M is lumped or banded, the change to the standard eigenproblem can be accomplished without much increase in computing time. The characteristic polynomial of Equation (5) can be represented as:

$$P(\lambda) = \det(A - \lambda I) = a_{2n}\lambda^{2n} + a_{2n-1}\lambda^{2n-1} + \cdots + a_1\lambda + a_0 = \sum_{i=0}^{2n} a_i\lambda^i = 0, \tag{7}$$

where λ is a complex value and a_i $(i = 0,1,\ldots,2n)$ are the real coefficients.

The coefficient of the characteristic polynomial

Chen suggested a numerically stable algorithm to obtain the coefficients of the characteristic polynomial of a real square matrix. According to his algorithm some given matrix A :

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,2n-1} & a_{1,2n} \\ a_{21} & a_{22} & \cdots & a_{2,2n-1} & a_{2,2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2n-1,1} & a_{2n-1,2} & \cdots & a_{2n-1,2n-1} & a_{2n-1,2n} \\ a_{2n,1} & a_{2n,2} & \cdots & a_{2n,2n-1} & a2_{n,2n} \end{bmatrix}, \tag{8}$$

can be transformed to $\bar{A}$:

$$\bar{A} = \begin{bmatrix} -a_{2n-1} & -a_{2n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{9}$$

by applying a sequence of Gauss-elimination like similarity transformations. When some numerical instability occurs during the transformations, the modified algorithm by Chen can be used. Since $\bar{A}$ was obtained by applying similar transformations to A, the eigenvalues and eigenvectors of both $\bar{A}$ and A are same. The characteristic polynomial of A, $P(\lambda) = \det(A - \lambda I)$ can be obtained by observing the transformed matrix $\bar{A}$, and the characteristic polynomial is:

$$P(\lambda) = \det(A - \lambda I) = \det(\bar{A} - \lambda I) = \lambda^{2n} + a_{2n-1}\lambda^{2n-1} + \cdots + a_1\lambda + a_0 = 0, \ (a_{2n} = 1). \tag{10}$$

The number of eigenvalues in an open unit disk.

Gleyse suggested a method of calculating the number of eigenvalues of a real polynomial inside an open unit disk by a determinant representation.

Let $P(\lambda) = \sum_{h=0}^{2n} a_h\lambda_h = 0$ (a_h is a real number) then the number of eigenvalues inside an open unit disk can be determined as:

$$N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}] \tag{11}$$

where N_λ is the number of eigenvalues in an open unit disk, $2n$ is the degree of the polynomial, $S[k_0, k_1, k_2, \cdots, k_{2n}]$ is the number of sign changes in the sequence k_i $(i = 0, 1, \cdots, 2n)$ and d_i $(i = 1, 2, \cdots, 2n)$ is the determinants(minors) of the leading principal submatrices of order i in the Schur-Cohn matrix T:

$$T = \begin{bmatrix} \ddots & & \\ & t_{ij} & \\ & & \ddots \end{bmatrix}, t_{ij} = \sum_{h=0}^{\min(i,j)} \left(a_{2n-i+h}a_{2n-j+h} - a_{i-h}a_{j-h}\right). \tag{12}$$

The process of calculating the number of eigenvalues using the above theorem requires the calculation of the characteristic polynomial of a given matrix, the construction of the Schur-Cohn matrix T and the calculation of the determinants (minors) of the leading principal submatrices of order i in the Schur-Cohn matrix T. The coefficients of the characteristic polynomial of a given matrix can be determined by applying previous Chen's algorithm, and by using these coefficients, each elements of the Schur-Cohn matrix can be obtained using Equation (12). The determinants(minors) of the leading principal submatrices of order i in the Schur-Cohn matrix T can be easily determined by applying LDL^T factorization of T, which is described in the following section.

Modified Sturm sequence property.

Gleyse's theorem considers only about the number of eigenvalues in an open unit disk. To apply his theorem for open disks of arbitrary radius ρ, substitute $\lambda = \rho\bar{\lambda}$ (ρ is an real number) to Equation (7), then:

$$P(\bar{\lambda}) = \bar{a}_{2n}\bar{\lambda}^{2n} + \bar{a}_{2n-1}\bar{\lambda}^{2n-1} + \cdots + \bar{a}_1\bar{\lambda} + \bar{a} = \sum_{i=0}^{2n} \bar{a}_i\bar{\lambda}^i = 0, \tag{13}$$

where $\bar{a}_i = a_i\rho^i \, (i = 0, 1, \cdots, 2n)$ are modified coefficients.

Using the modified coefficients $\bar{a}_i \, (i = 0, 1, \cdots, 2n)$ in Equation (13), His theorem can be extended to calculate the number of eigenvalues in the open disks of arbitrary radius ρ. The calculation of $d_i \, (i = 1, \cdots, 2n)$ can be easily performed by the LDL^T factorization of the Schur-Cohn matrix T. If $T = LDL^T$, then:

$$T_i = L_i D_i L_i^T \tag{14}$$

where T_i is the leading principal submatrices of order i in the T, L_i is the leading principal submatrices of order i in the L and D_i is the leading principal submatrices of order i in the D. Therefore each $d_i \, (i = 1, \cdots, 2n)$ can be easily obtained as:

$$\begin{aligned} d_i &= \det(T_i) = \det(L_i D_i L_i^T) = \det(D_i) \\ &= d_{11} \times d_{22} \times \cdots \times d_{ii} = \prod_{h=1}^{i} d_{hh} \end{aligned} \tag{15}$$

Considering Equation (11), we only need to know the signs of each d_i because the unknown value of $S[1, d_1, d_2, \cdots, d_{2n}]$ depends only on sign changes of each $d_i \, (i = 1, \cdots, 2n)$, and from Equation (15) the signs of each d_i can be determined from the number of negative elements of each diagonal elements of D_i, which is very similar to Strum sequence property for undamped systems.

A NUMERICAL EXAMPLE

To show the effectiveness of the proposed method, a simple spring-mass-damper system that has the exact analytical eigenvalues is considered to verify that the proposed method can exactly calculate the number of eigenvalues in the open disk of arbitrary radius for the eigenproblem with the damping matrix.

Simple Spring-Mass-Damper System

The finite element discretization of the system results in a diagonal mass matrix, a tridiagonal damping and stiffness matrices of the following forms:

$$M = mI \tag{16}$$

$$C = \alpha M + \beta K \tag{17}$$

$$K = k\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \tag{18}$$

where α and β are the damping coefficients of the Rayleigh damping. The analytical solutions can be resulted through following relationships:

$$\lambda_{2i-1,2i} = -\xi_i\omega_i \pm j\omega_i\sqrt{1-\xi_i^2} \quad \text{for } i = 1,\ldots,n \tag{19}$$

$$\xi_i = \frac{1}{2}\left(\frac{\alpha}{\omega_i} + \beta\omega_i\right) \tag{20}$$

$$\omega_i = 2\sqrt{\frac{m}{k}}\sin\frac{2i-1}{2n+1}\frac{\pi}{2} \tag{21}$$

where ω_i and ξ_i are the undamped natural frequency and modal damping ratio, respectively.

A system with order 10 is used in analysis. k and m are 1, and the coefficients, α and β, of the Rayleigh damping are 0.05 and 0.5, respectively. All the eigenvalues and their radius from the origin in the complex plane are as in Table 1.

Table 1

The calculated eigenvalues, coefficients $\bar{a}_i$, diagonal elements d_{ii} of D

I	Eigenvalue(λ)		Radius ($\rho=\lvert\lambda\rvert$)	$\rho = 1.005\lvert\lambda_{16}\rvert$ = 1.8109		$\rho = 1.005\lvert\lambda_{18}\rvert$ = 1.9207		$\rho = 1.005\lvert\lambda_{20}\rvert$ = 1.9875	
	Real	Imag.		$\bar{a}_i$	d_{ii}	$\bar{a}_i$	d_{ii}	$\bar{a}_i$	d_{ii}
0	-	-	-	3.079e-04	-	2.055e-04	-	1.618e-00	-
1	-0.0306	-0.1463	0.1495	4.322e-03	1.962e+03	3.059e-03	9.206e+03	2.492e-00	2.239e+04
2	-0.0306	0.1463	0.1495	8.065e-02	1.962e+03	6.055e-02	9.206e+03	5.104e-00	2.239e+04
3	-0.0745	-0.4388	0.4450	6.252e-01	1.962e+03	4.979e-01	9.206e+03	4.343e-00	2.239e+04
4	-0.0745	0.4388	0.4450	4.147e-00	1.962e+03	3.503e-00	9.206e+03	3.161e-00	2.239e+04
5	-0.1585	-0.7133	0.7307	1.947e-00	1.959e+03	1.745e+01	9.204e+03	1.629e+01	2.239e+04
6	-0.1585	0.7133	0.7307	7.601e+01	1.948e+03	7.222e+01	9.187e+03	6.979e+01	2.237e+04
7	-0.2750	-0.9614	1.0000	2.373e+02	1.808e+03	2.391e+02	8.977e+03	2.391e+02	2.212e+04
8	-0.2750	0.9614	1.0000	6.251e+02	1.657e+03	6.681e+02	8.511e+03	6.913e+02	2.135e+04
9	-0.4137	-1.1763	1.2470	1.374e+03	8.707e+02	1.558e+03	6.368e+03	1.668e+03	1.773e+04
10	-0.4137	1.1763	1.2470	2.581e+03	7.447e+02	3.104e+03	5.248e+03	3.439e+03	1.476e+04
11	-0.5624	-1.3540	1.4661	4.111e+03	7.311e+02	5.242e+03	1.799e+03	6.011e+03	6.872e+03
12	-0.5624	1.3540	1.4661	5.610e+03	6.757e+02	7.588e+03	1.459e+03	9.002e+03	5.379e+03
13	-0.7077	-1.4932	1.6525	6.495e+03	-1.056e+02	9.318e+03	1.331e+02	1.144e+04	9.257e+02
14	-0.7077	1.4932	1.6525	6.395e+03	-6.349e-00	9.731e+03	1.098e+02	1.236e+04	7.403e+02
15	-0.8368	-1.5959	1.8019	5.261e+03	2.020e-00	8.491e+03	9.891e+00	1.116e+04	3.517e+01
16	-0.8368	1.5959	1.8019	3.592e+03	1.688e-00	6.149e+03	8.390e+00	8.365e+03	2.792e+01
17	-0.9381	-1.6651	1.9111	1.960e+03	-3.303e-00	3.558e+03	-3.412e+00	5.008e+03	2.735e-00
18	-0.9381	1.6651	1.9111	8.325e+02	-2.537e-00	1.603e+03	-2.312e+00	2.335e+03	2.111e-00
19	-1.0028	-1.7046	1.9777	2.446e+02	3.976e-00	4.996e+02	1.065e+00	7.529e+02	1.622e-00
20	-1.0028	1.7046	1.9777	4.429e+01	2.866e-00	9.595E+01	7.366e+00	1.496e+02	1.208e-00

In this example the checking process performed for open disks of three different radius. First two cases are for checking in between eigenvalues. Their location is selected considering the relative distances between two adjacent eigenvalues except for a conjugate eigenvalues, and two eigenvalues with smaller distances from the next are selected. The third is for checking the number of all the eigenvalues of the system. The radius of the open disk is should be only a bit larger than the selected eigenvalue to ensure that the next eigenvalue is not within the open disk. Jung et al. recommended 1.005 times the magnitude of the largest known eigenvalue. In this example, therefore, the radius ρ of the disk is chosen by 1.005 times the magnitude of the largest eigenvalue($\rho = 1.005|\lambda|$). For each cases, the calculated coefficient of the characteristic polynomial $\bar{a}_i$ and diagonal element d_{ii} are as in Table II. Using the signs of d_{ii}, the number of eigenvalues for each cases are calculated as followings:

Case 1: $\rho = 1.005|\lambda_{16}| = 1.8109$, $N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}]$, $N_\lambda = 20 - 4 = 16$

Case 2: $\rho = 1.005|\lambda_{18}| = 1.9207$, $N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}]$, $N_\lambda = 20 - 2 = 18$

Case 3: $\rho = 1.005|\lambda_{20}| = 1.9875$, $N_\lambda = 2n - S[1, d_1, d_2, \cdots, d_{2n}]$, $N_\lambda = 20 - 0 = 20$

Referring to Table 1, the number of eigenvalues which is inside open disks of radius $\rho = 1.005|\lambda_{16}| = 1.8109$, $\rho = 1.005|\lambda_{18}| = 1.9207$ and $\rho = 1.005|\lambda_{20}| = 1.9875$ are 16, 18 and 20 which are exactly agree with the calculated values. As seen from this result, therefore, we verify that the proposed method can exactly check the number of eigenvalues inside some open disk of arbitrary radius.

CONCLUSIONS

A technique of calculating the number of eigenvalues inside an open disk of arbitrary radius was given. The technique is based on Chen's algorithm and Glyese's theorem and can be used to check the missed eiegnvalues for the eigenproblem with damping matrix. By analyzing the numerical examples, it is verified that the proposed method can exactly check the number of eigenvalues for distinct or multiple eigenvalues for damped systems. The technique by Jung et al. should find the variation of arguments of complex numbers along a predefined path. Therefore, a large number of checking points sholud be used to obtain accurate result. However, the proposed method can exactly find the number of eigenvalues by performing the factorization process only once. In result, much effort in finding the number of eigenvalue of larger structures with damping matrix can be eliminated by the proposed method.

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