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SIMPLIFIED ALGEBRAIC METHOD FOR COMPUTING EIGENPAIR SENSITIVITIES OF DAMPED SYSTEMS

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ABSTRACT
A simplified method for the eigenpair sensitivities of damped systems is presented. This approach employs a reduced equation to determine the sensitivities of eigenpairs of the damped vibratory systems with distinct eigenvalues. The derivatives of eigenpairs are obtained by solving an algebraic equation with a symmetric coefficient matrix of (n+1) by (n+1) dimension where n is the number of degree of freedom. This is an improved method of the previous work of Lee and Jung. Two equations are used to find eigenvalue derivatives and eigenvector derivatives in their paper. A significant advantage of this approach over Lee and Jung is that one algebraic equation newly developed is enough to compute such eigenvalue derivatives and eigenvector derivatives. Simulation results indicate that the new method is highly efficient in determining the sensitivities of eigenpairs of the damped vibratory systems with distinct eigenvalues.

KEYWORDS
Sensitivity analysis, eigenpair, eigenproblem, eigensystem.

INTRODUCTION

Methods for computing the derivatives of natural frequencies and the corresponding mode shapes have been studied in the past 30 years. Finding the derivatives of eigenpairs are essential to determine the sensitivity of dynamic responses of the physical systems. For structural design, these eigenpair derivatives are used to optimize the natural frequencies and the mode shapes of structures by varying design parameters.

Rudisill & Chu (1975) presented an algebraic method for the eigenvector derivatives. This method is restricted to the case of non-repeated eigenvalue problem and this method has an asymmetric coefficient matrix. Nelson (1976) solved the same problem; His technique requires only the knowledge of eigenvector to be differentiated and is recommended as an efficient solver for calculating the mode shape derivatives. This method is limited to the distinct eigenvalue problem too. Because of its complicated algorithm, programming code is lengthy and clumsy. Ojalvo (1988) extended Nelson's method to the multiple eigenvalue problem. Mills-Curren (1988) and Dailey (1989) modified Ojalvo's work. Because those methods are based on Nelson's, their algorithms are so complicated too. In addition to these techniques, modal method (Murthy & Haftka 1988, Lim & Junkins 1987) and its modified one (Wang 1985, Liu *et al.* 1987) approximate the mode shape derivatives by a linear combination of mode shapes. It takes lots of computing time when these approaches are used due to a large number of modes. Additionally, we have an iterative method (Andrew 1978, Tan 1986 and Lee & Jung 1996), whose drawback is its inaccurate solution. Lee & Jung (1997a, b) studied an algebraic method for computing the eigenvalue and eigenvector derivatives of general matrix with non-repeated and repeated eigenvalues. This approach is very efficient and simple and it needs the only corresponding eigenvalue and eigenvector. Furthermore, not only an exact solution is obtained but also numerical stability is proved in their method.

A number of the prescribed methods can be applied to the damped systems; Hallquist (1976) proposed a method for determining the effects of mass modification in viscously damped systems. Recently Zimoch (1987) presented a sensitivity analysis method, which is applied to conservative ones as well as non-conservative systems. It, however, may be restricted to mechanical systems (lumped systems) having only distinct eigenvalues. In other words, implementing this method to the systems with multiple eigenvalues is difficult. Lee & Jung's method (1997a, b) is extended to the damped systems by Lee *et al.* (1999a, b).

In this paper, an improved method over Lee *et al.* (1999a, b) is studied with reduction of the number of equations for the eigenpair derivatives. The eigenvalue derivatives are obtained apart from the eigenvector derivatives solving two equations in Lee *et al.* (1999a, b). But the eigenpair derivatives are found by solving one modified equation in this paper. Therefore, the FLOPS are reduced for computing to get the eigenpair derivatives while maintaining the advantages of Lee *et al.* (1999a, b). The algebraic equation of the proposed method is efficiently solved by the LDL^T type decomposition method. If the derivatives of stiffness, mass and damping matrices can be obtained analytically, the proposed method can find the exact eigenpair derivatives.

The proposed method and its stability proof are made in the second section of this paper. The results are illustrated with a numerical example in the next section.

PROPOSED METHOD

Derivation of the proposed method

Consider a multi-degree-of-freedom damped system described as

$$(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})\phi = \mathbf{0}\,, \tag{1}$$

where **M**, **C** and **K** are the matrices of mass, damping and stiffness respectively, and $n \times n$ symmetric matrices. **M** is positive definite and **K** is positive definite or semi-positive definite. λ and ϕ are the eigenvalue and eigenvector and both are complex values in general. To determine the eigenvalues and eigenvectors of damped system, one can use state-space-form described by $2n$ –dimensional eigenvalue problem as

$$\begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix} = \lambda \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix}, \tag{2}$$

which can be written conveniently as

$$\mathbf{Az} = \lambda\mathbf{Bz}\,, \tag{3}$$

where

$$\mathbf{A} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \; B = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \text{ and } \; \mathbf{z} = \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix}. \tag{4}$$

The solutions of the complex eigenvalue problem Eqn.3. can be found and they are distinct and conjugate. The eigenvectors are normalized such as

$$\mathbf{z}_j^T\mathbf{B}\,\mathbf{z}_j = \begin{Bmatrix} \phi_j \\ \lambda_j\phi_j \end{Bmatrix}^T \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \phi_j \\ \lambda_j\phi_j \end{Bmatrix} = 1, \tag{5}$$

and arranging Eqn.5. gives

$$\phi_j^T(2\lambda_j\mathbf{M}+\mathbf{C})\phi_j=1. \tag{6}$$

Suppose that all eigenpairs and matrices $\partial K/\partial p$, $\partial M/\partial p$ and $\partial C/\partial p$ are known, and all eigenvlaues are different, where p is a design parameter. To obtain the derivatives of eigenvalue and eigenvector, the differentials of the eigenvalue problem and normalization condition are used.

Differentiating Eqn. 1. gives

$$(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K})\frac{\partial\phi_j}{\partial p}+(2\lambda_j\mathbf{M}+\mathbf{C})\phi_j\frac{\partial\lambda_j}{\partial p}=-\left(\lambda_j^2\frac{\partial\mathbf{M}}{\partial p}+\lambda_j\frac{\partial\mathbf{C}}{\partial p}+\frac{\partial\mathbf{K}}{\partial p}\right)\phi_j. \tag{7}$$

Differentiating Eqn 6. gives

$$\phi_j^T(2\lambda_j\mathbf{M}+\mathbf{C})\frac{\partial\phi_j}{\partial p}+\phi_j^T\mathbf{M}\phi_j\frac{\partial\lambda_j}{\partial p}=-\frac{1}{2}\phi_j^T\left(2\lambda_j\frac{\partial\mathbf{M}}{\partial p}+\frac{\partial\mathbf{C}}{\partial p}\right)\phi_j. \tag{8}$$

Because the unknown or interested values are $\partial\phi_j/\partial p$ and $\partial\lambda_j/\partial p$, Eqn. 7. and Eqn.8 can be combined as single matrix form as Eqn. 9. In Lee *et al.*, two equation is needed for the derivatives of the eigenvalue and eigenvector, because $\partial\lambda_j/\partial p$ is calculated apart from $\partial\phi_j/\partial p$ by pre-multiplying ϕ_j^T at each side of Eqn. 7.

$$\begin{bmatrix}\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K} & (2\lambda_j\mathbf{M}+\mathbf{C})\phi_j\\ \phi_j^T(2\lambda_j\mathbf{M}+\mathbf{C}) & \phi_j^T\mathbf{M}\phi_j\end{bmatrix}\begin{Bmatrix}\dfrac{\partial\phi_j}{\partial p}\\ \dfrac{\partial\lambda_j}{\partial p}\end{Bmatrix}=\begin{Bmatrix}-\left(\lambda_j^2\dfrac{\partial\mathbf{M}}{\partial p}+\lambda_j\dfrac{\partial\mathbf{C}}{\partial p}+\dfrac{\partial\mathbf{K}}{\partial p}\right)\phi_j\\ -\dfrac{1}{2}\phi_j^T\left(2\lambda_j\dfrac{\partial\mathbf{M}}{\partial p}+\dfrac{\partial\mathbf{C}}{\partial p}\right)\phi_j\end{Bmatrix}. \tag{9}$$

But the proposed method finds $\partial\phi_j/\partial p$ and $\partial\lambda_j/\partial p$ at once by solving the single matrix, one algebraic equation, composed of the differentials of the eigenvalue problem and normalization condition.

The proposed method has the characteristics of not only finding exact solutions, having the numerical stability and having a symmetric coefficient matrix which are those of Lee *et al.*, but also being more efficient for the time required to calculate the eigenpair derivatives because of being composed of more simple equation.

Numerical stability of the proposed method

To prove that the coefficient matrix $\mathbf{A}^{\#}$ of Eqn. 9. is nonsingular, introduce the nonsingular square matrix $\mathbf{Y}$, $\det(\mathbf{Y})\neq 0$. Matrix $\mathbf{Y}$ is used in the determinant property, $\det(\mathbf{Y}^T\mathbf{A}^{\#}\mathbf{Y})=\det(\mathbf{Y}^T)\det(\mathbf{A}^{\#})\det(\mathbf{Y})$. If $\det(\mathbf{Y}^T\mathbf{A}^{\#}\mathbf{Y})\neq 0$, the determinant of $\mathbf{A}^{\#}$ is nonzero because the determinant of $\mathbf{Y}$ is nonzero. Nonsingular matrix $\mathbf{Y}$ is assumed as

$$\mathbf{Y}=\begin{bmatrix}\Psi & \mathbf{0}\\ \mathbf{0} & 1\end{bmatrix}, \tag{10}$$

where $\Psi=[\psi_1\ \psi_2\ \cdots\ \psi_{n-1}\ \phi_j]$, ϕ_j is the j th eigenvector of the system and ψ 's are arbitrary vectors to be independent of ϕ_j. Ψ is a $n\times n$ matrix, $\mathbf{Y}$ is a $(n+1)\times(n+1)$ matrix.

Pre- and post-multiplying $\mathbf{Y}^T$ and $\mathbf{Y}$ to $\mathbf{Y}^T$ yields

$$
\begin{aligned}
\mathbf{Y}^T\mathbf{A}^{\#}\mathbf{Y} &= \begin{bmatrix} \Psi & 0 \\ \mathbf{0} & 1 \end{bmatrix}^T \left[\begin{array}{c|c} \lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K} & (2\lambda_j\mathbf{M}+\mathbf{C})\phi_j \\ \hline \phi_j^T(2\lambda_j\mathbf{M}+\mathbf{C}) & \phi_j^{\ T}\mathbf{M}\phi_j \end{array}\right] \begin{bmatrix} \Psi & 0 \\ \mathbf{0} & 1 \end{bmatrix} \\
&= \left[\begin{array}{c|c} \Psi^T(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K})\Psi & \Psi^T(2\lambda_j\mathbf{M}+\mathbf{C})\phi_j \\ \hline \phi_j^T(2\lambda_j\mathbf{M}+\mathbf{C})\Psi & \phi_j^{\ T}\mathbf{M}\phi_j \end{array}\right]
\end{aligned}
\tag{11}
$$

The last column and row of the matrix $\mathbf{\Psi}^T(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K})\mathbf{\Psi}$ is zero because of ϕ_j which is the last column of Ψ. That is

$$
\Psi^T(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K})\Psi = \begin{bmatrix} \widetilde{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \tag{12}
$$

where $\widetilde{\mathbf{A}}$ is a nonzero $(n-1)\times(n-1)$ submatrix. And λ_j is a distinct eigenvalue of the system, therefore the matrices $\mathbf{\Psi}^T(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K})\mathbf{\Psi}$ of order n have a rank $n-1$, they are singular. But the submatrix $\widetilde{\mathbf{A}}$ of order $n-1$ has full rank $n-1$, and it is nonsingular, $\det(\widetilde{\mathbf{A}})\neq 0$.
By normalization condition the last elements of the column vector $\Psi^T(2\lambda_j\mathbf{M}+\mathbf{C})\phi_j$ and the row vector $\phi_j^T(2\lambda_j\mathbf{M}+\mathbf{C})\Psi$ are unity.

$$
\Psi^T(2\lambda_j\mathbf{M}+\mathbf{C})\phi_j = \begin{Bmatrix} \widetilde{\mathbf{b}} \\ 1 \end{Bmatrix} \quad \text{and} \quad \phi_j^T(2\lambda_j\mathbf{M}+\mathbf{C})\Psi = \begin{Bmatrix} \widetilde{\mathbf{b}} \\ 1 \end{Bmatrix}^T, \tag{13}
$$

where $\widetilde{\mathbf{b}}$ is nonzero vector. Substituting Eqn. 12. and Eqn. 13. into Eqn. 11 gives

$$
\mathbf{Y}^T\mathbf{A}^{\#}\mathbf{Y} = \begin{bmatrix} \widetilde{\mathbf{A}} & \mathbf{0} & \widetilde{\mathbf{b}} \\ \mathbf{0} & 0 & 1 \\ \widetilde{\mathbf{b}}^T & 1 & \phi_j^{\ T}\mathbf{M}\phi_j \end{bmatrix}. \tag{14}
$$

Applying the determinant property of partitioned matrix gives that

$$
\det(\mathbf{Y}^T\mathbf{A}^{\#}\mathbf{Y}) = \det\begin{bmatrix} 0 & 1 \\ 1 & \phi_j^{\ T}\mathbf{M}\phi_j \end{bmatrix} \det\left(\widetilde{\mathbf{A}} - \begin{bmatrix} \mathbf{0} & \widetilde{\mathbf{b}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & \phi_j^{\ T}\mathbf{M}\phi_j \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \widetilde{\mathbf{b}}^T \end{bmatrix} \right), \tag{15}
$$

where

$$
\begin{bmatrix} \mathbf{0} & \widetilde{\mathbf{b}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & \phi_j^{\ T}\mathbf{M}\phi_j \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \widetilde{\mathbf{b}}^T \end{bmatrix} = 0 \quad \text{and} \quad \det\begin{bmatrix} 0 & 1 \\ 1 & \phi_j^{\ T}\mathbf{M}\phi_j \end{bmatrix} = -1. \tag{16}
$$

Therefore rearranging equation (15) gives

$$
\det(\mathbf{Y}^T\mathbf{A}^{\#}\mathbf{Y}) = -\det(\widetilde{\mathbf{A}}) \neq 0. \tag{17}
$$

The determinant of $\mathbf{A}^{\#}$ thus is not equal to zero, in other words, the matrix $\mathbf{A}^{\#}$ is nonsingular.

NUMERICAL EXAMPLE

The efficiency and exactness of Lee & Jung's method are verified in numerical examples. In this section, the results by Lee & Jung's method and by proposed method with a cantilever beam with multidampers, are compared, for the case of the distinct natural frequencies. Pentium 120 having CPU capacity 120Mhz with RAM 40Mega is used for computation.

Cantilever beam with multi-lumped dampers

A cantilever beam with 100 elements and 100 multi-lumped dampers is considered as shown in Figure 1. The number of nodes is 101 and each node has tow degrees of freedom (y-translation, z-rotation); the total number of degrees of freedom is 200. For this example, Young's modulus (1000), the mass density (1) and damper (0.3), cross-section inertia (1) and cross-section area (1) are used. The length of the beam $5m$.

Non-proportional damping combined lumped damping with rayleigh damping is used. The Rayleigh coefficients is $\alpha = \beta = 0.01$. The design parameter is the depth of beam, t.

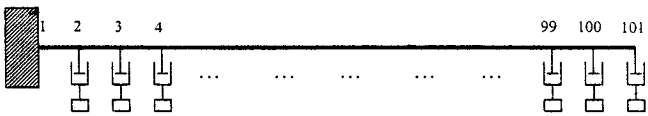

Fig. 1. Cantilever beam with multi-lumped damper

Now the system matrices M, C, K and their derivatives $\partial M/\partial t$, $\partial C/\partial t$, $\partial K/\partial t$ are known values. Some sensitivity results are represented in Table 1 and Table 2.

As shown in Table 1 and Table 2, we can see that the results of Lee & Jung's method and the proposed method are the same. It is exact solution. The analysis time is 143.03 seconds for Lee & Jung's method and 106.44 seconds for the proposed method to find 200 eigenpair derivatives. The analysis time of the proposed method is less than that of Lee & Jung's method by 25.6%.

Table 1. The lowest twenty eigenvalue and their derivatives

Mode Number	Eigenvalue	Eigenvalue derivative (Lee&Jung's method)	Eigenvalue derivative (Proposed method)
1	-1.0023e-02-0.0803e+00i	-3.5930e-03 -9.8965e-01i	-3.5930e-03 -9.8965e-01i
2	-1.0023e-02+0.0803e+00i	-3.5930e-03 +9.8965e-01i	-3.5930e-03 +9.8965e-01i
3	-1.7454e-01-1.8435e+01i	-4.1931e-02 -2.4175e+00i	-4.1931e-02 -2.4175e+00i
4	-1.7454e-01+1.8435e+01i	-4.1931e-02+2.4175e+00i	-4.1931e-02 +2.4175e+00i
5	-7.9353e-01-3.9686e+01i	-1.3377e-01 -3.4644e+00i	-1.3377e-01 -3.4644e+00i
6	-7.9353e-01+3.9686e+01i	-1.3377e-01+3.4644e+00i	-1.3377e-01 +3.4644e+00i
7	-1.8708e+00 -6.1033e+01i	-2.0046e-01 -3.3550e+00i	-2.0046e-01 -3.3550e+00i
8	-1.8708e+00 -6.1033e+01i	-2.0046e-01+3.3550e+00i	-2.0046e-01 +3.3550e+00i
9	-3.4027e+00 -8.2340e+01i	-2.4881e-01 -3.0748e+00i	-2.4881e-01 -3.0748e+00i
10	-3.4027e+00 -8.2340e+01i	-2.4881e-01+3.0748e+00i	-2.4881e-01 +3.0748e+00i

Table 2. Some components of the first eigenvector and its derivatives

DOF Number	Eigenvector	Eigenvector derivative (Lee&Jung's method)	Eigenvector derivative (Proposed method)
1	-3.4119e-05 -3.4110e-05i	1.1362e-05 +1.1353e-05i	1.1362e-05 +1.1353e-05i
2	-1.3620e-03 -1.3616e-03i	4.5337e-04 +4.5301e-04i	4.5337e-04 +4.5301e-04i
3	-1.3592e-04 -1.3589e-04i	4.5225e-05 +4.5189e-05i	4.5225e-05 +4.5189e-05i
4	-2.7075e-03 -2.7068e-03i	9.0004e-04 +8.9934e-04i	9.0004e-04 +8.9934e-04i
5	-3.0459e-04 -3.0451e-04i	1.0125e-04 +1.0117e-04i	1.0125e-04 +1.0117e-04i
6	-4.0364e-03 -4.0353e-03i	1.3400e-03 +1.3390e-03i	1.3400e-03 +1.3390e-03i
7	-5.3929e-04 -5.3915e-04i	1.7911e-04 +1.7897e-04i	1.7911e-04 +1.7897e-04i
8	-5.3486e-03 -5.3473e-03i	1.7733e-03 +1.7719e-03i	1.7733e-03 +1.7719e-03i
9	-8.3918e-04 -8.3896e-04i	2.7847e-04 +2.7826e-04i	2.7847e-04 +2.7826e-04i
10	-6.6442e-03 -6.6425e-03i	2.1999e-03 +2.1983e-03i	2.1999e-03 +2.1983e-03i

CONCLUSIONS

To calculate the derivatives of eigenvalue and eigenvector, two equations were used in the previous work. But, by unifying the two equations by one, a simple algorithm is developed and simulated for the systems with non-repeated eigenvalues.

The proposed method is an improvement of Lee *et al:* An exact solution is obtained and the numerical stability is proved as in Lee *et al* with a simplified algorithm. Additionally, CPU time to compute the eigenpair derivatives is remarkably reduced due to the simplification of algorithm in the proposed method as compared with Lee *et al.*

Table 3. CPU time spent on the calculation of the first 200 eigenpairs

Method	Operations	CPU time(sec)
Lee & Jung's Method	$\frac{\partial \lambda_j}{\partial p} = -\phi_j^T\left(\lambda_j^2\frac{\partial \mathbf{M}}{\partial p}+\lambda_j\frac{\partial \mathbf{C}}{\partial p}+\frac{\partial \mathbf{K}}{\partial p}\right)\phi_j$	28.12
	$\mathbf{A}^* = \begin{bmatrix} \lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K} & (2\lambda_j\mathbf{M}+\mathbf{C})\phi_j \\ \phi_j^T(2\lambda_j\mathbf{M}+\mathbf{C}) & 0 \end{bmatrix}$	35.01
	$\mathbf{f}_j = \begin{Bmatrix} -(2\lambda_j\mathbf{M}+\mathbf{C})\phi_j\frac{\partial \lambda_j}{\partial p} - \left(\lambda_j^2\frac{\partial \mathbf{M}}{\partial p}+\lambda_j\frac{\partial \mathbf{C}}{\partial p}+\frac{\partial \mathbf{K}}{\partial p}\right)\phi_j \\ -\frac{1}{2}\phi_j^T\left[2\left(\frac{\partial \lambda_j}{\partial p}\mathbf{M}+\lambda_j\frac{\partial \mathbf{M}}{\partial p}\right)+\frac{\partial \mathbf{C}}{\partial p}\right]\phi_j \end{Bmatrix}$	29.56
	$\begin{Bmatrix} \frac{\partial \phi_j}{\partial p} \\ 0 \end{Bmatrix} = [\mathbf{A}^*]^{-1}\mathbf{f}_j$	50.34
	Total	**143.03**
Proposed Method	$\mathbf{A}^* = \begin{bmatrix} \lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K} & (2\lambda_j\mathbf{M}+\mathbf{C})\phi_j \\ \phi_j^T(2\lambda_j\mathbf{M}+\mathbf{C}) & \phi_j^T\mathbf{M}\phi_j \end{bmatrix}$	31.52
	$\bar{\mathbf{f}}_j = \begin{Bmatrix} -\left(\lambda_j^2\frac{\partial \mathbf{M}}{\partial p}+\lambda_j\frac{\partial \mathbf{C}}{\partial p}+\frac{\partial \mathbf{K}}{\partial p}\right)\phi_j \\ -\frac{1}{2}\phi_j^T\left(2\lambda_j\frac{\partial \mathbf{M}}{\partial p}+\frac{\partial \mathbf{C}}{\partial p}\right)\phi_j \end{Bmatrix}$	26.60
	$\begin{Bmatrix} \frac{\partial \phi_j}{\partial p} \\ \frac{\partial \lambda_j}{\partial p} \end{Bmatrix} = [\mathbf{A}^*]^{-1}\bar{\mathbf{f}}_j$	48.32
	Total	**106.44**

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