MATRIX POWER LANCZOS METHOD AND ITS APPLICATION TO THE EIGENSOLUTION OF STRUCTURES

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Abstract

This paper applies the matrix-powered Lanczos method developed in quantum physics to the eigensolution in structural dynamics. In structural problems, the power technique can be applied to the dynamic matrix. The convergence of the modified Lanczos method using the power of dynamic matrix is better than that of the conventional Lanczos method. By analyzing numerical examples, the effectiveness of the modified Lanczos method is verified and the optimal power of dynamic matrix is presented.

Introduction

Lanczos method (Lanczos, 1950) has been known to be very efficient for the eigensolution of structures. To improve the Lanczos method many researchers have studied a variety of procedures. Erricson and Ruhe (1980) have used shifting techniques to accelerate the Lanczos algorithm. Smith et al. (1993) have accelerated the Lanczos method through an implicitly restarted technique. Gambolati and Putti (1994) employed the preconditioned conjugate gradient scheme in the Lanczos method. Kim and Lee (1999) introduced a Lanczos-based algorithm for the solution of eigenproblems for non-classically damped structures. In the fields of quantum physics, Grosso et al. (1993) modified the Lanczos algorithm with power of operator to obtain the eigenstate of quantum systems. Similar power technique is found in accelerated subspace iteration method for structural dynamics (Lam and Bertolini, 1994; Qian and Dhatt, 1995; Wang and Zhou, 1999). While, the modified Lanczos method using the power technique is not applied to structural dynamics yet. This paper applies it to the eigensolution of structural dynamics. In structural eigenproblem, the power technique can be applied to the matrix $\mathbf{K}^{-1}\mathbf{M}$ which is called dynamic matrix (Clough and Penzien, 1993). The modified Lanczos method using the power of dynamic matrix (*Matrix power Lanczos method* is proposed as the name.) can accelerate the convergence of the conventional Lanczos method. Four numerical examples are presented to verify the effectiveness of matrix power Lanczos method. The optimal power of dynamic matrix in the method is also presented through numerical examples.

Matrix power Lanczos method

In the fields of quantum physics, Grosso et al. (1993) modified Lanczos recursion by introducing the second power of operator to accelerate the convergence as follows;

$$b_{n+1}f_{n+1} = (H - E_t)^2 f_n - a_n f_n - b_n f_{n-1} \tag{1}$$

where H is a given operator, f is basis functions, a and b are coefficients and n is Lanczos step number. E_t is trial energy which corresponds to shift in structural dynamics. The concept of power technique in (1) can be applied to the eigenproblem in structural dynamics. The eigenproblem of structure frequently encountered in structural dynamics can be expressed as

$$\mathbf{K}\phi_i = \lambda_i \mathbf{M}\phi_i \quad (i = 1, 2, 3, \cdots, n) \tag{2}$$

where $\mathbf{M}$ and $\mathbf{K}$ are symmetric mass and stiffness matrices of order n, respectively. λ_i and ϕ_i are the ith eigenvalue and associated eigenvector of the system. To get the solution of (2), Lanczos schemed Ritz bases vectors through Gram-Schmidt orthogonalization of Krylov sequence as follows (Hughes, 1987);

$$\mathbf{x}_{i+1} = (\mathbf{K}_\mu^{-1}\mathbf{M})^i \mathbf{x}_0 - \sum_{j=1}^{i} \upsilon_j \mathbf{x}_j \tag{3}$$

where $\mathbf{x}_0$ is a starting vector, $\mathbf{x}_j$ is jth Lanczos vector, υ_j is the component of $\mathbf{v}_i$ along $\mathbf{x}_j$, $\mathbf{K}_\mu = \mathbf{K} - \mu\mathbf{M}$ and μ is shift. The concept of power technique can be applied to the dynamic matrix in (3), then following modified Gram-Schmidt orthogonalization can be introduced.

$$\mathbf{x}_{i+1} = ((\mathbf{K}_\mu^{-1}\mathbf{M})^\delta)^i \mathbf{x}_0 - \sum_{j=1}^{i} \upsilon_j \mathbf{x}_j \tag{4}$$

Where δ is positive integer. (4) means that an approximated eigenvector, whose number of iteration is δi, is contained in $(i+1)$ Lanczos vectors. Whereas, in (3), $(i+1)$ Lanczos vectors contain an approximated eigenvector whose number of iterations is i. Therefore, (4) gives a better solution than (3). From (4), modified Lanczos recursion can be derived as

$$\tilde{\mathbf{x}}_i = (\mathbf{K}_\mu^{-1}\mathbf{M})^\delta \mathbf{x}_i - \alpha_i \mathbf{x}_i - \beta_{i-1}\mathbf{x}_{i-1} \tag{5}$$

where α_i and β_i are scalar coefficients obtained by

$$\alpha_i = \mathbf{x}_i^T \mathbf{M} (\mathbf{K}_\mu^{-1}\mathbf{M})^\delta \mathbf{x}_i \ , \ \beta_i = (\tilde{\mathbf{x}}_i^T \mathbf{M} \tilde{\mathbf{x}}_i)^{1/2} \tag{6}$$

then the next Lanczos vector is

$$\mathbf{x}_{i+1} = \tilde{\mathbf{x}}_i / \beta_i \tag{7}$$

With a set of Lanczos vectors, $\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \cdots \mathbf{x}_q]$, we can obtain the tridiagonalized standard eigenproblem of reduced order $q << n$

$$\mathbf{T}\tilde{\phi}_i = (1/(\lambda_i - \mu)^\delta)\tilde{\phi}_i \quad (i = 1, 2, 3, \cdots, q) \tag{8}$$

where

$$\mathbf{T} = \mathbf{X}^T\mathbf{M}(\mathbf{K}_\mu^{-1}\mathbf{M})^\delta\mathbf{X} = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & & \ddots & & \\ & & & \alpha_{q-1} & \beta_{q-1} \\ & & & \beta_{q-1} & \alpha_q \end{bmatrix} \tag{9}$$

The Lanczos algorithm is subjected to loss of orthogonality of Lanczos vectors due to round-off errors. In this paper, full reorthogonalization process (Bathe, 1996) is used to retain the orthogonality of the Lanczos vectors. The number of total operations for matrix power Lanczos algorithm is

$$\begin{aligned} N_{total} &= (1/2)nm^2 + (q^2 + 4q\delta + 5q + 3/2)nm \\ &+ \{(3/2)q^2 + q\delta + (17/2)q\}n + 10q^2 + q + \sum_{j=2}^{q} 6js_j \end{aligned} \tag{10}$$

where n is system order, m halfband-width, q the number of calculated Lanczos vectors and s_j the number of iterations of jth step in QR iteration for the eigenvalues of tridiagonal system.

Numerical examples

A simple spring-mass system with 100 DOFs (Chen, 1993), a plane framed structure (Bathe and Wilson, 1972), a three-dimensional frame structure (Bathe and Wilson, 1972) and a three-dimensional building frame (Kim and Lee, 1999) are analyzed to verify the effectiveness of matrix power Lanczos method. With the predetermined error norm of 10^{-6}, the number of operations for calculating desired eigenpairs is compared. To examine the optimal power of dynamic matrix, numerical examples are analyzed with varying power of dynamic matrix. System matrices of a simple spring-mass system are

$$\mathbf{M} = \mathbf{I}\ ,\ \mathbf{K} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \tag{11}$$

The geometric configurations and the material properties of a plane framed structure, a three-dimensional frame structure and a three-dimensional building frame are shown in Figs. 1 ~ 3.

Some results are shown in Table 1 and Fig. 4. The 1st power (δ = 1) corresponds to the conventional Lanczos method. Table 1 and Fig. 4 show that the convergence of matrix power Lanczos method is better than that of the conventional Lanczos method. However, in some cases, high matrix power causes failure in convergence due to the numerical instability.

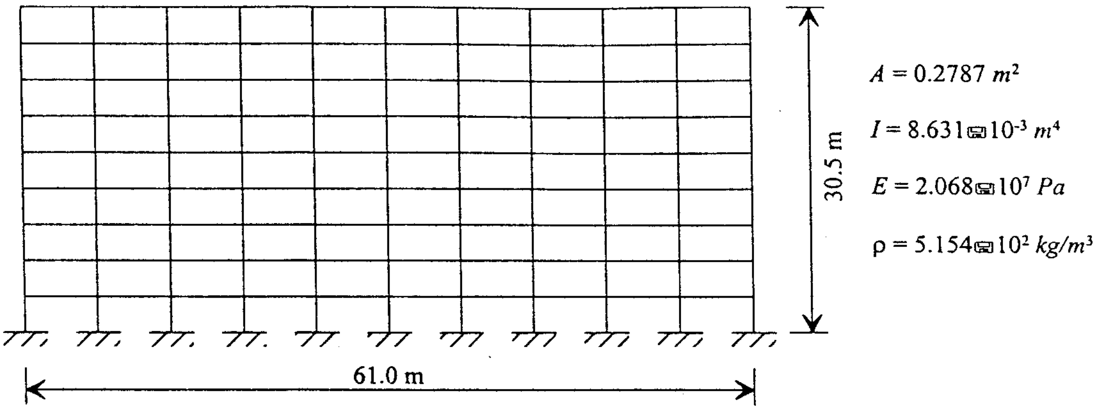
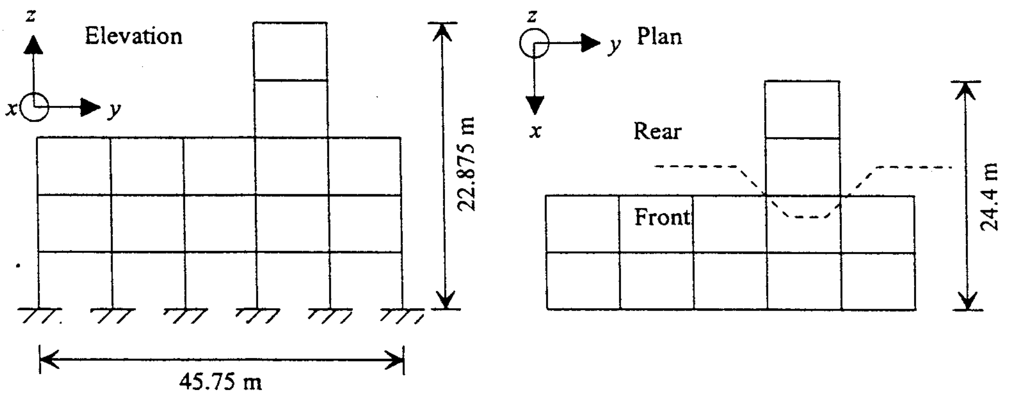


Fig. 1. Plane framed structure (DOFs: 330)



Column in front building : $A = 0.2787\ m^2$, $I = 8.631\times10^{-3}\ m^4$
Column in Rear building : $A = 0.3716\ m^2$, $I = 10.789\times10^{-3}\ m^4$
All beams into x-Diretion : $A = 0.1858\ m^2$, $I = 6.473\times10^{-3}\ m^4$
All beams into y-Diretion : $A = 0.2787\ m^2$, $I = 8.631\times10^{-3}\ m^4$

$E = 2.068\times10^{7}\ Pa$
$\rho = 5.154\times10^{2}\ kg/m^3$

Fig. 2. Three-dimensional frame structure (DOFs: 468)

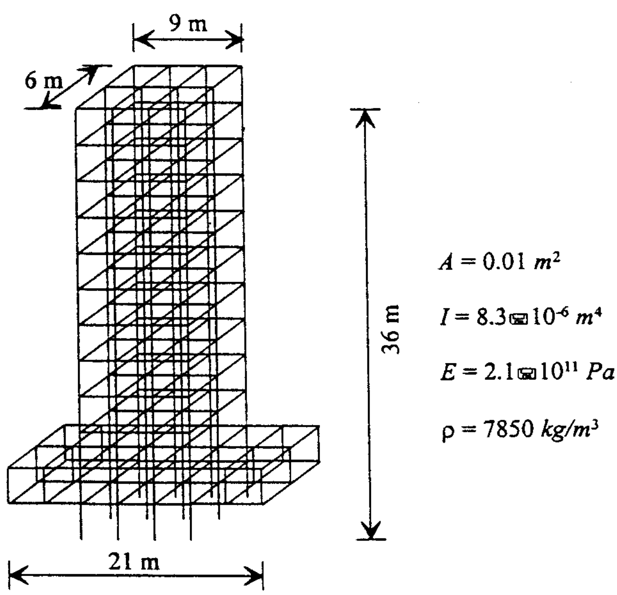
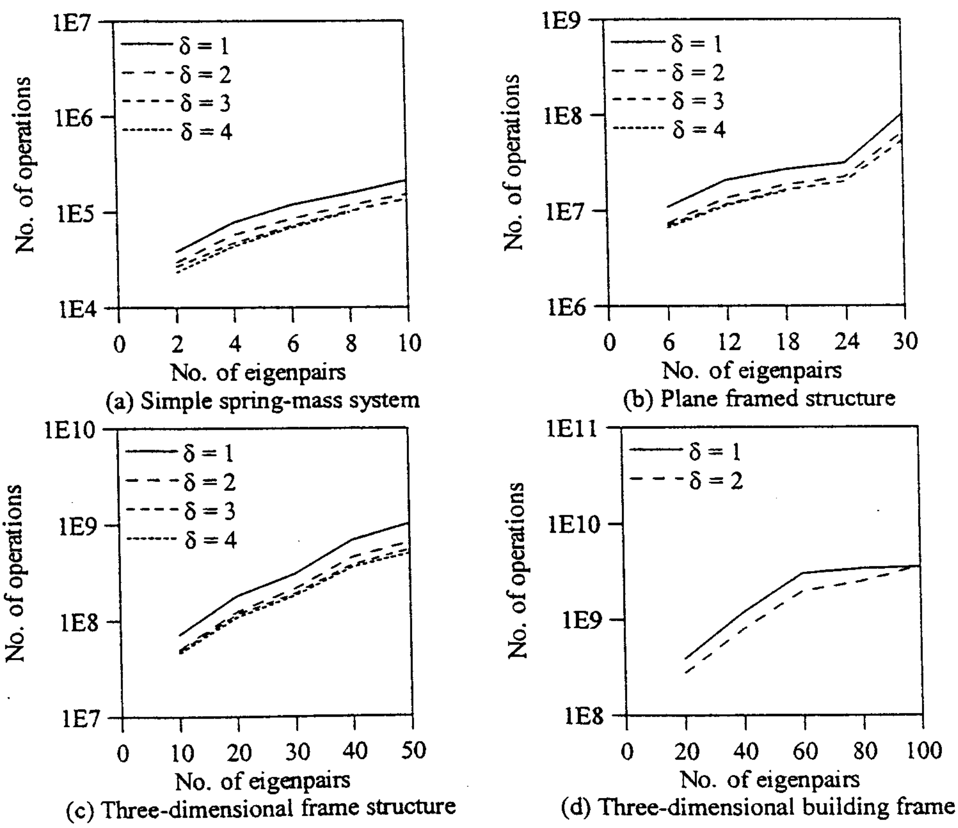


Fig. 3. Three-dimensional building frame (DOFs: 1008)

Table 1. Number of operations for calculating desired eigenpairs

Structure	No. of eigenpairs	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$
Simple spring-mass system	2	38663	29823	26954	23653
	4	78922	58529	47567	44122
	6	120458	85712	73040	69391
	8	157649	117587	103055	99550
	10	214729	154418	138122	*
Plane framed structure	6	10908273	7429050	7072452	6633536
	12	20855865	13578945	11688377	11237625
	18	27029145	18676209	16508507	16047093
	24	31581179	22516533	20164797	*
	30	102944376	65994807	54112986	*
Three-dimensional frame structure	10	71602154	50687925	48705515	46214349
	20	181780512	124269611	116680070	108715163
	30	307269560	215884077	192064376	182518601
	40	684162222	453454527	378770940	356596304
	50	1024104917	656188310	553972908	504420108
Three-dimensional building frame	20	395079020	278717178	*	*
	40	1196316954	801878160	*	*
	60	3045578295	1993108128	*	*
	80	3398746793	2509125474	*	*
	100	3536190824	3625240574	*	*

* : Failure in convergence due to numerical instability



(a) Simple spring-mass system

(b) Plane framed structure

(c) Three-dimensional frame structure

(d) Three-dimensional building frame

Fig. 4. Number of operations for calculating desired eigenpairs

5. Conclusions

This paper applies a power technique to the Lanczos method for the eigenproblem solution of structures. The characteristics of matrix power Lanczos method by the numerical results from examples are summarized as follows:

(1) Since the power of dynamic matrix in matrix power Lanczos method can reduce the number of required Lanczos vectors, the convergence of matrix power Lanczos method is better than that of the conventional Lanczos method.

(2) The optimal power of dynamic matrix that reduces the number of operations and gives numerically stable solution in matrix power Lanczos method is the second power.

Acknowledgements

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References

Bathe, K. J. (1996), *Finite element procedures*, Prentice-Hall, Inc.

Bathe, K. J. and Wilson, E. L. (1972), "Large eigenvalue problems in dynamic analysis," *J. Engrg. Mech., ASCE*, **98**, 1471–1485.

Chen, H. C. (1993), "Partial eigensolution of damped structural systems by Arnoldi's method," *Earthquake Engrg.* and *Struct. Dynamics*, **22**, 63–74.

Clough, R. W. and Penzien, J. (1993), *Dynamics of structures*, McGraw-Hill, Inc.

Ericsson, T. and Ruhe, A. (1980), "The spectral transformation Lanczos method for the numerical solution of large sparse generalized symmetric eigenvalue problems," *Math. Comp.*, **35**(152), 1251–1268.

Gambolati, G. and Putti, M. (1994), "A comparison of Lanczos and optimization methods in the partial solution of sparse symmetric eigenproblems," *Int. J. Numer. Methods in Engrg.*, **37**, 605-621.

Grosso, G., Martinelli, L. and Parravicini, G. P. (1993), "A new method for determining excited states of quantum systems," *Nuovo Cimento D*, **15**(2-3), 269–277.

Hughes, T. J. R. (1987), *The finite element method; Linear static and dynamic finite element analysis*, Prentice-Hall, Inc.

Kim, M. C. and Lee, I. W. (1999), "A computationally efficient algorithm for the solution of eigenproblems for large structures with non-proportional damping using Lanczos method," *Earthquake Engrg.* and *Struct. Dynamics*, **28**, 157–172.

Lam, Y. C. and Bertolini, A. F. (1994), "Acceleration of the subspace iteration method by selective repeated inverse iteration," *FE Anal. Design*, **18**, 309–317.

Lanczos, C. (1950), "An iteration method for the solution of the eigenvalue problem of linear differential and integral operators," *J. Res. Natl. Bur. Stand.*, **45**(4), 255–282.

Qian, Y. Y. and Dhatt, G. (1995), "An accelerated subspace method for generalized eigenproblems," *Comp. and Struct.*, **54**(6), 1127–1134.

Smith, H. A., Sorensen, D. C. and Singh, R. K. (1993), "A Lanczos-based technique for exact vibration analysis of skeletal structures," *Int. J. Numer. Methods in Engrg.*, **36**, 1987–2000.

Wang, X. and Zhou, J. (1999), "An accelerated subspace iteration method for generalized eigenproblems," *Comp. and Struct.*, **71**, 293–301.