Wave Propagation and Mode Localization in Simply Supported Multispan Beams with Couplers on Supports

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Abstract: The influences of couplers on wave propagation and mode localization in simply supported multispan beams with couplers consisting of lumped rotational stiffness and mass on supports are studied. A transfer matrix equation governing the vibrational wave propagation in the simply supported multispan beams with couplers is newly derived and simplified. The eigenvalue of the simplified transfer matrix shows that the larger stiffness or the larger mass of the coupler makes the internal coupling between spans weaker and so it makes the system more sensitive to mode localization. As the wave frequency or the eigenvalues of the system increases, the mass effect is increased while the stiffness effect is decreased. In a case considering the large stiffness and mass at the same time, there is a region with relatively wider passbands and narrower stop bands having small attenuation rates and the normal modes in it become delocalized ones. As an example structure, a simply supported two span beam with a coupler at the midspan is considered.

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Introduction

Many engineering structures are made of identical substructures connected by couplers, shaping into periodic structures. For some periodic structures, the presence of small irregularities breaking the assumption of structural identity may significantly affect mode shapes of the structures and may be a cause of mode localization. Mode localization results in that the vibration energy of the mode is confined to specific regions, and so the structure may suffer unpredicted local damages and the performance of a motion controller of the structure might decrease abruptly.

In solid-state physics, the localization phenomenon of an electron field in a disordered solid was first observed by Anderson (1958) who shared the 1977 Nobel Prize in physics for his work. Hodges (1982) was the first to recognize that the wave localization may occur in the disordered periodic structures and it leads to mode localization. In many works (Bendiksen 1987; Cornwell and Bendiksen 1987; Pierre and Dowell 1987; Pierre et al. 1987; Cornwell and Bendiksen 1989a,b; Bouzit and Pierre 1992; Lust et al. 1993; Langley 1995), various structures have been considered and many methods have been proposed to discuss the characteristics of mode localization.

It is well known that, under a condition of weak coupling by a coupler’s stiffness, the mode shapes undergo dramatic changes to become strongly localized when a small disorder is introduced. To date, however, little attention has been paid to the influences of the mass of a coupler on coupling strength and mode localization.

The present study is an attempt to prove that the mass, as well as the stiffness, of the coupler exerts important influences upon mode localization and weak coupling conditions. This work is an extended work of Kim and Lee (1999) in which a simple lumped system was considered. In “Formulations” of this paper, a transfer matrix equation governing the vibrational wave propagation in the simply supported multispan beams with couplers is derived and simplified for theoretical study. In “Wave Propagation,” using the eigenvalue of the simplified transfer matrix, the influences of the coupler consisting of a lumped rotational stiffness and a lumped rotational mass on coupling strength and on wave propagation are discussed. “Mode Localization of Two Span Beam” is an example verifying the influences of a coupler on mode localization, where a simply supported two span beam with a coupler at the midspan is considered.

Formulations

In this section a transfer matrix equation governing the wave propagation in simply supported multispan beams with couplers on supports is formulated. Fig. 1 shows the structure considered. Each span has the flexural rigidity EI, the mass density ρ, and the length 2Lₘ. The jth coupler consists of a lumped rotational stiffness Kₖ and a lumped rotational mass Jₖ. The coordinate system of the jth span is shown in Fig. 2. An eigenvalue problem representing the free vibration of the jth span is as follows:

\[
\frac{d^4 y_j(x_j)}{dx_j^4} - \lambda^4 y_j(x_j) = 0
\]  

where \(\lambda\) is an eigenvalue of the system having natural frequency \(\omega_n\), and has the following relationship:

\[
\lambda^4 = \omega_n^4 \frac{m}{EI}
\]
The general solution of Eq. (1) may be assumed as

$$y_j(x_j) = A_j \cosh \lambda x_j + B_j \sinh \lambda x_j + C_j \cos \lambda x_j + D_j \sin \lambda x_j$$

(3)

where the constants $A_j$, $B_j$, $C_j$, and $D_j$ may be determined by applying boundary conditions of the $j$th span. By applying the simple support conditions of each span, such as $y_j(-L_j) = 0$ and $y_j(L_j) = 0$, Eq. (3) is reduced to

$$y_j(x_j) = C_j \left( \cos \lambda x_j - \frac{\cos \lambda L_j}{\cosh \lambda L_j} \cos \lambda x_j \right)$$

$$+ D_j \left( \sin \lambda x_j - \frac{\sin \lambda L_j}{\sinh \lambda L_j} \sin \lambda x_j \right)$$

(4)

Applications of the slope and moment continuity conditions for any consecutive spans $j$ and $j+1$ in Eq. (4) and simplification yield a single matrix equation as follows:

$$T_{j+1} \mathbf{y}_j = \mathbf{y}_{j+1}$$

(5)

where

$$\mathbf{y}_j = \begin{bmatrix} C_j \\ D_j \end{bmatrix} \quad \text{and} \quad \mathbf{y}_{j+1} = \begin{bmatrix} C_{j+1} \\ D_{j+1} \end{bmatrix}$$

(6)

$$T_{j+1} = \begin{bmatrix} 1 & \sin 2\lambda L_{j+1} \\ \sinh 2\lambda L_{j+1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

(7)

$$T_{11} = \psi_{j+1} \cos \lambda L_j - \psi_j \sin \lambda L_j + \frac{K_{j+1} - \omega_n^2 J_{j+1}}{2\lambda EI} \psi_{j+1}$$

$$T_{12} = \psi_{j+1} \sin \lambda L_j + \psi_j \sin \lambda L_j + \frac{K_{j+1} - \omega_n^2 J_{j+1}}{2\lambda EI} \psi_{j+1}$$

$$T_{21} = -\psi_{j+1} \cos \lambda L_j - \psi_j \cos \lambda L_j + \frac{K_{j+1} - \omega_n^2 J_{j+1}}{2\lambda EI} \psi_{j+1}$$

$$T_{22} = -\psi_{j+1} \sin \lambda L_j + \psi_j \cos \lambda L_j + \frac{K_{j+1} - \omega_n^2 J_{j+1}}{2\lambda EI} \psi_{j+1}$$

(8a, 8b, 8c)

and

$$\theta_j = \sin \lambda L_j + \tanh \lambda L_j \cos \lambda L_j$$

and

$$\psi_j = \cos \lambda L_j - \coth \lambda L_j \sin \lambda L_j$$

(9)

Eq. (5) is a transfer matrix equation associated with the propagation of vibration between adjacent two spans. Applications of the moment equilibrium conditions defined at the both ends of the structure in Eq. (4) yield the two boundary conditions as follows:

$$\mathbf{b}_1^T \mathbf{y}_1 = 0 \quad \text{and} \quad \mathbf{b}_n^T \mathbf{y}_n = 0$$

(10)

where the boundary condition vectors are

$$\mathbf{b}_1 = \begin{bmatrix} -\cos \lambda L_1 - \frac{K_1 - \omega_n^2 J_1}{2\lambda EI} \theta_1 \\ \sin \lambda L_1 - \frac{K_1 - \omega_n^2 J_1}{2\lambda EI} \psi_1 \end{bmatrix}$$

and

$$\mathbf{b}_{n+1} = \begin{bmatrix} -\cos \lambda L_n - \frac{K_n - \omega_n^2 J_{n+1}}{2\lambda EI} \theta_n \\ -\sin \lambda L_n - \frac{K_n - \omega_n^2 J_{n+1}}{2\lambda EI} \psi_n \end{bmatrix}$$

(11)

Combining the transfer matrix equation, Eq. (5), and the boundary conditions, Eq. (10), gives the characteristic equation of the system as follows:

$$f(\lambda) = \mathbf{b}_{n+1}^T \prod_{k=1}^{n} T_{k+1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{b}_1 = 0$$

(12)

By solving the characteristic equation, we get the eigenvalues of the system. Repeated application of Eq. (5) starting with an assumption of $\mathbf{y}_1 = [0, -1; 1, 0] \mathbf{b}_1$ and the eigenvalue given by solving Eq. (12) yields all the state vectors $\mathbf{y}_j$’s of the spans, and finally we get the eigenvectors of the system.

Numerical study using the transfer matrix equation, the boundary conditions, and the characteristic equation may give many interesting results. However, since there are hyperbolic terms, it is almost impossible to study theoretically with those equations and conditions. Therefore it is desirable to make them simple for the theoretical study. Fortunately, since the terms of hyperbolic tangent and hyperbolic cotangent in the equations converge to unity very quickly with the increase of $\lambda L_j$ and $\lambda L_{j+1}$, under the assumptions of $\lambda L_j > 1$ and $\lambda L_{j+1} > 1$ the transfer matrix can be simplified as

$$T_{j+1} = \mathbf{R}(-\lambda L_{j+1}) C\left(c_{j+1}\right) \mathbf{R}(-\lambda L_j)$$

(13)

where $\mathbf{R}(-\lambda L_{j+1})$ and $\mathbf{R}(-\lambda L_j)$ = rotation matrices such as
\[
R(-\lambda L_{j+1}) = \begin{bmatrix}
\cos \lambda L_{j+1} & \sin \lambda L_{j+1} \\
\sin \lambda L_{j+1} & \cos \lambda L_{j+1}
\end{bmatrix}
\]

and

\[
R(-\lambda L_{j}) = \begin{bmatrix}
\cos \lambda L_{j} & \sin \lambda L_{j} \\
\sin \lambda L_{j} & \cos \lambda L_{j}
\end{bmatrix}
\]

and \(C(c_{j+1})\) has the form of

\[
C(c_{j+1}) = c_{j+1} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} + \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]

where

\[
c_{j+1} = 1 + \frac{1}{\lambda L} [\tilde{K}_{j+1} - \tilde{J}_{j+1} (\lambda L)^4]
\]

As you can see, \(c_{j+1}\) is a function of \(\lambda L\) and the stiffness and mass of the coupler, where \(L\) = arbitrary reference half-span length, and \(\tilde{K}_{j+1}\) and \(\tilde{J}_{j+1}\) = dimensionless stiffness and mass

\[
\tilde{K}_{j+1} = \frac{1}{2} \frac{K_{j+1} L}{EI} \quad \text{and} \quad \tilde{J}_{j+1} = \frac{1}{2} \frac{J_{j+1}}{mL^3}
\]

Under the same assumptions of \(\lambda L_{1} > 1\) and \(\lambda L_{n} > 1\), the boundary condition vectors, Eq. (11), may be simplified as

\[
b_{1} = -R(-\lambda L_{1}) \begin{bmatrix} c_{1} \\ c_{1} - 1 \end{bmatrix} \quad \text{and} \quad b_{n+1} = R(\lambda L_{n}) \begin{bmatrix} -c_{n+1} \\ c_{n+1} - 1 \end{bmatrix}
\]

**Wave Propagation**

The behavior of energy-carrying waves can be characterized by the eigenvalues of the transfer matrix. In this study, the transfer matrix has two eigenvalues given by solving

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**Fig. 4.** Influences of a coupler on the passband/stop band structure, where \(\gamma\) = real part of propagation constant, and \(\sigma\) = imaginary part. The dotted lines (⋅⋅⋅) represent the case of \(\tilde{K} = 0.0, \tilde{J} = 0.0\); (a) ⋅⋅⋅⋅⋅⋅ \(\tilde{K} = 3.0, \tilde{J} = 0.0\); ⋅⋅⋅⋅⋅⋅ \(\tilde{K} = 10.0, \tilde{J} = 0.0\); (b) ⋅⋅⋅⋅⋅⋅ \(\tilde{K} = 0.0, \tilde{J} = 0.03\); ⋅⋅⋅⋅⋅⋅ \(\tilde{K} = 0.0, \tilde{J} = 0.1\); (c) ⋅⋅⋅⋅⋅⋅ \(\tilde{K} = 3.0, \tilde{J} = 0.03\); ⋅⋅⋅⋅⋅⋅ \(\tilde{K} = 3.0, \tilde{J} = 0.1\); and (d) ⋅⋅⋅⋅⋅⋅ \(\tilde{K} = 10.0, \tilde{J} = 0.03\); ⋅⋅⋅⋅⋅⋅ \(\tilde{K} = 10.0, \tilde{J} = 0.1\).

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**Fig. 5.** Simply supported two-span beam with a coupler at the mid-support.
where $\lambda_u$=eigenvalue of the transfer matrix and $I=2\times2$ identity matrix. Considering the case of $L_{j+1}=L_j$, it is easy to show that the eigenvalues have the form of $\lambda_u=e^{\pm j\mu}$ where $\mu$=wave propagation constant. Generally the wave propagation constant is a complex number, such as $\mu=\gamma+j\sigma$. The real part of the propagation constant, $\gamma$, =rate of exponential attenuation of the wave amplitude, and the imaginary part, $\sigma$, =difference in phase between the motion of adjacent spans. Fig. 3 shows $\gamma$ and $\sigma$ of the case of $L_{j+1}=L_j$ and $\tilde{K}_{j+1}=\tilde{J}_{j+1}=0$ ($c_{j+1}=1$) as functions of $\lambda L/\pi$. The regions of $\gamma=0$ are passbands and the regions of $\gamma\neq0$ are stop bands. The waves in passbands propagate without attenuation in magnitude. In the stop bands only attenuated standing waves can exist. In what follows, the influences of the stiffness and mass of a coupler on the passband/stop band structure are studied.

Using the simplified transfer matrix with $L=(L_{j+1}+L_j)/2$ leads the eigenvalues to

\[\text{det}[T_{j+1} - \lambda_u I] = 0\]  \hspace{1cm} (19)
Table 1. Dimensionless Eigenvalues, $\lambda L/\pi$, of the System Considered in Fig. 6(a)

<table>
<thead>
<tr>
<th>Mode number</th>
<th>$K = 3.0, J = 0.0$</th>
<th>$K = 10.0, J = 0.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.00$</td>
<td>$0.568$</td>
<td>$0.600$</td>
</tr>
<tr>
<td>$\alpha = 0.03$</td>
<td>$0.564$</td>
<td>$0.635$</td>
</tr>
<tr>
<td>$\alpha = 0.05$</td>
<td>$0.558$</td>
<td>$0.646$</td>
</tr>
<tr>
<td>$\lambda_n = \sqrt{(c_{j+1}^2 + 1) \cos(2\lambda L + \phi_{j+1})}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pm \sqrt{(c_{j+1}^2 + 1) \cos^2(2\lambda L + \phi_{j+1}) - 1}$ (20)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[ \phi_{j+1} = \tan^{-1} \frac{1}{c_{j+1}} ] (21)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Dimensionless Eigenvalues, $\lambda L/\pi$, of the System Considered in Fig. 6(b)

<table>
<thead>
<tr>
<th>Mode number</th>
<th>$K = 0.0, J = 0.0$</th>
<th>$K = 10.0, J = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.00$</td>
<td>$0.491$</td>
<td>$0.400$</td>
</tr>
<tr>
<td>$\alpha = 0.03$</td>
<td>$0.490$</td>
<td>$0.469$</td>
</tr>
<tr>
<td>$\alpha = 0.05$</td>
<td>$0.488$</td>
<td>$0.468$</td>
</tr>
</tbody>
</table>

Table 3. Dimensionless Eigenvalues, $\lambda L/\pi$, of the System Considered in Fig. 6(c)

<table>
<thead>
<tr>
<th>Mode number</th>
<th>$K = 3.0, J = 0.03$</th>
<th>$K = 3.0, J = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.00$</td>
<td>$0.565$</td>
<td>$0.557$</td>
</tr>
<tr>
<td>$\alpha = 0.03$</td>
<td>$0.561$</td>
<td>$0.553$</td>
</tr>
<tr>
<td>$\alpha = 0.05$</td>
<td>$0.555$</td>
<td>$0.549$</td>
</tr>
</tbody>
</table>

Table 4. Dimensionless Eigenvalues, $\lambda L/\pi$, of the System Considered in Fig. 6(d)

<table>
<thead>
<tr>
<th>Mode number</th>
<th>$K = 10.0, J = 0.03$</th>
<th>$K = 10.0, J = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.00$</td>
<td>$0.599$</td>
<td>$0.597$</td>
</tr>
<tr>
<td>$\alpha = 0.03$</td>
<td>$0.590$</td>
<td>$0.589$</td>
</tr>
<tr>
<td>$\alpha = 0.05$</td>
<td>$0.581$</td>
<td>$0.580$</td>
</tr>
</tbody>
</table>

Mature consideration with Eqs. (16) and (21), and the two conditions defined by Eqs. (22) and (23) may yield interesting results. The larger stiffness of coupler results in smaller $C_\delta$ in the lower frequency range, and the larger mass of coupler results in smaller $C_\delta$ in the higher frequency range. Fig. 4 shows the influences of $K_{j+1}$ and $J_{j+1}$ on the passband/stop band structures, and the variations of coupling strength $C_\delta$ as functions of $\lambda L/\pi$. As predicted, the larger $K_{j+1}$ or $J_{j+1}$ leads to smaller $C_\delta$ causing narrower passbands and wider stop bands with a larger rate of exponential attenuation. The influence of the mass increases with the increase of $\lambda L$ while the influence of the stiffness decreases. A certain $\lambda L$ leading to $C_\delta = \pm \pi/2$ (or $c_{j+1} = 0$) is always in a passband.

Since mode localization occurs when a mode on the edge of a passband is moved to the stop band by disordering the periodic structure, the weak internal coupling causing narrow passbands and large attenuation rates is a precondition of drastic occurrence of mode localization. Therefore the larger stiffness of a coupler

mode localization. Generally, the coupling strength is measured by $1/K_{j+1}$ but it is not suited to the cases of considering $K_{j+1}$ and $J_{j+1}$ at the same time. In this paper, the angle $\phi_{j+1}$ is proposed as a measure of coupling strength $C_\delta$. If all the couplers in a multi-span beam have the identical properties, one can rewrite $\phi_{j+1}$ without the subscript, and the coupling strength representing the strengths of internal couplings between spans of the structure can be defined as

$$C_\delta = \phi$$ (24)
may increases the sensitivity to mode localization of the modes in the lower frequency range, and the larger mass may do so in the higher frequency range. In the next section, these influences will be verified with a simple example.

**Mode Localization of Two Span Beam**

In this example study, the influences of a lumped rotational stiffness and a lumped rotational mass constituting a coupling of the simply supported two span beam shown in Fig. 5 on mode localization are studied. The lengths of the spans are assumed as \( L_1 = L(1 + \alpha) \) and \( L_2 = L(1 - \alpha) \) where \( L \) = reference half-span length and \( \alpha = \text{disturbance breaking the periodicity of the structure.} \) The cases considered are consistent with those in the previous section and Fig. 4.

In this example study, to facilitate discussion for mode localization of multispans beams, a measure of the degree of mode localization, \( D_L \), is defined as

\[
D_L = \frac{(\bar{y}_1 + \bar{y}_2)^2}{\bar{y}_1^2 + \bar{y}_2^2}
\]

(25)

where \( \bar{y}_i \) = absolute value of the maximum amplitude associated with the ith span. The degree of mode localization is to be unity, \( D_L = 1 \), when the mode is extremely localized, and to be zero, \( D_L = 0 \), when the mode is not localized at all.

Localization curves in Fig. 6(a) show the typical behavior of mode localization and the influences of \( \tilde{K} \) on mode localization. Degrees of mode localization increase with the increase of \( \alpha \) and \( \tilde{K} \). The influence of \( \tilde{J} \) differs from that of \( \tilde{K} \) and the localization curves in Fig. 6(b) show it. Degrees of mode localization increase with the increases of \( \alpha \), mode number, and \( \tilde{J} \). That is, the mass of the coupler makes the structure sensitive to mode localization, and the mass effect is more pronounced in the higher modes. Localization curves plotted in Figs. 6(c and d) show the combined influences of \( \tilde{K} \) and \( \tilde{J} \). The degree of mode localization decreases with the increase of mode number until certain modes, but after that it increases abruptly with mode number. The modes for which mode localization does not occur or is very weak although structural disturbances are severe are referred to as delocalized modes. These delocalization phenomena are more dramatic in cases of the larger stiffness and larger mass, Fig. 6(d). Considering the modes far from the delocalized ones, the behavior of mode localization is governed by \( \tilde{K} \) for lower modes but by \( \tilde{J} \) for higher modes. As you can see in Figs. 4 and 5 in the previous section and from Tables 1–4, a mode in a passband and near the edge of the band is moved to stop band and localized by introducing the span length disturbance \( \alpha \). The delocalized modes are near the points of \( C_S = \pm \pi/2 \).

**Concluding Remarks**

In this work, the influences of the stiffness and the mass of the coupler on wave propagation and mode localization have been studied. Some important conclusions drawn in the course of this work can be summarized as follows:

1. The larger stiffness of the coupler results in the weaker internal coupling and the influence of stiffness decreases with the increase of wave frequency;

2. The larger mass of the coupler results in the weaker internal coupling and the influence of mass increases abruptly with the increase of wave frequency;

3. The larger stiffness of the coupler increases the sensitivity to mode localization of the normal modes of a multispans beam in the lower frequency range;

4. The larger mass of the coupler increases the sensitivity to mode localization of the normal modes of a multispans beam in the higher frequency range; and

5. Considering the stiffness and the mass of the coupler at the same time, there is a certain frequency region having a relatively wider passband caused by the countervailing property between the stiffness and the mass. This leads to delocalization phenomena in some normal modes.

The results of this work may provide a very useful guide for designing a structure insensitive to mode localization.

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**References**


