박 사 학 위 논 문

부분공간 반복법 개선 및 비비례감쇠 구조물의 누락된 고유치 검사 기법 개발

정 형 조 (鄭 炯 朝)

토 목 공 학 과

한 국 과 학 기 술 원

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Improvement of Subspace Iteration Method and Development of Technique for Checking Missed Eigenvalues of Structures with Nonproportional Damping

Improvement of Subspace Iteration Method and Development of Technique for Checking Missed Eigenvalues of Structures with Nonproportional Damping

Advisor : Professor **In-Won Lee**

by

Hyung Jo Jung

Department of Civil Engineering
Korea Advanced Institute of Science and Technology

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Approved by

Professor **In Won Lee**
Major Advisor

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정 형 조

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ABSTRACT

In order to obtain the exact dynamic response of a structure or to avoid the resonant response of a structure, it is required to develop a solution method that can evaluate the desired eigenvalues and the corresponding eigenvectors and a technique that can check whether the desired eigenpairs are indeed calculated without any missed ones or not. This dissertation presents a numerically stable eigenproblem solution method by improving the subspace iteration method with shift and a technique for checking the missed eigenvalues of structures with nonproportional damping.

The subspace iteration method has hitherto been known to be very efficient for solving large eigenproblems. A major difficulty of the conventional subspace iteration method using shifting technique is that a shift very close to an eigenvalue cannot be used due to the singularity problem, resulting in slower convergence. In this study, the above singularity problem has been solved by introducing side conditions without sacrifice of convergence. The proposed method is always nonsingular even if a shift is on a distinct eigenvalue or multiple ones. This is one of the significant characteristics of the proposed method. The nonsingularity is proved analytically. The convergence of the proposed method is at least equal to that of the conventional subspace iteration method with shift, which is also proved analytically, and the operation counts of the above two methods are almost the same when a large number of eigenpairs are required. To show the effectiveness of the proposed method, four cases are analyzed. They are two structures

with distinct eigenvalues such as a three-dimensional frame structure and a simply supported rectangular plate structure and two structures with multiple eigenvalues such as a three-dimensional frame structure with a symmetric cross-section and a simply supported square plate structure.

Most of the eigenvalue analysis methods such as the subspace iteration method and the Lanczos method may miss some eigenpairs in the required ones, because the above eigenvalue analysis methods do not calculate the complete eigenvector set of a structure but the lowest small portion of this set. For the practical eigenvalue analysis method, a technique to check the missed eigenvalues must be included. In the case of the undamped or proportionally damped system (i.e., in the eigenvalue analysis without damping), the missed eigenvalues can be checked by using the Sturm sequence property, and most of the eigenvalue analysis methods include the technique for checking the missed eigenvalues using the Sturm sequence property. However, in the case of the nonproportionally damped system such as the soil-structure interaction problem, the structural control problem and composite structures (i.e., in the eigenvalue analysis including damping), a technique for checking the missed eigenvalues has not been developed yet. In this study, various mathematical properties such as the extension of the Sturm sequence property, the Gerschgorin theorem, the Routh-Hurwitz criterion and the argument principle have been reviewed thoroughly. And then, a technique for checking the missed eigenvalues of structures with nonproportional damping is proposed by applying the argument principle. To verify the effectiveness of the proposed method, two numerical examples such as a simple spring-mass-damper system and a three-dimensional frame structure with concentrated dampers are considered.

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CHAPTER 1

INTRODUCTION

1.1 Background

The dynamic analysis of structures is important in many fields of engineering. The dynamic analysis is divided into two solution methods, the direct integration method and the mode superposition method. The choice for one method or the other is determined only by their numerical effectiveness. For loading of a relatively short duration such as impulse loading (i.e., for a few time steps), the direct integration method is usually the most effective method, because the number of operations required in the direct integration are directly proportional to the number of time steps used in the analysis. On the other hand, for loading of long duration such as in an earthquake (i.e., for many time steps), the mode superposition method is the appropriate method.

If the dynamic analysis for a structure is performed by the mode superposition method, the mode shapes (i.e., the eigenvectors) of the structure are needed to transform the equilibrium equations into more effective form for the direct integration. To get the mode shapes, the eigenvalue analysis of the structure must be first performed. The eigenvalue analysis of the structure is the most time consuming step in the mode superposition method. To efficiently obtain the dynamic response for a structure, therefore, an efficient and numerically stable eigenproblem solution method is needed.

The eigenproblem solution methods widely used in these days are the subspace iteration method (Bathe, 1971; Bathe and Wilson, 1972), the determinant search method (Bathe and Wilson, 1973; Gupta, 1973), the Lanczos method (Lanczos, 1950; Bathe, 1996) and so on. These eigenvalue analysis methods calculate only the lowest eigenvalues and the corresponding eigenvectors, because it usually gives a very good approximation to the exact dynamic response although the modal transformation matrix is

not composed of the complete eigenvector set of the structure but the lowest small portion of this set. However, the above eigenvalue analysis methods may miss some eigenpairs in the required ones. If the modal transformation process is performed with the lowest incomplete eigenvectors, the results of the dynamic analysis may not be a good approximation to the exact response. In this case, the missed eigenpairs should be extracted to construct the lowest complete eigenvector set, before the modal transformation step is performed.

As seen from the above discussion, to get the exact dynamic response by performing the mode superposition method, an efficient and numerically stable eigenproblem solution method and a technique to exactly check the missed eigenpairs are required.

1.2 Literature Review

1.2.1 Eigenvalue Analysis Methods

Many solution methods have been developed for the eigenvalue analysis. These methods can be divided into the approximate solution methods and the exact solution methods.

The approximate solution methods consist of static condensation method, dynamic condensation method, Rayleigh-Ritz method, component mode synthesis and Lanczos method. These methods are essentially techniques for reducing the size of a system of equations. The reduction of a system of equations eventually leads to a loss in accuracy of a solution. However, the advantage of lessened computational effort for a solution sometimes may compensate for the loss in accuracy. Moreover, an approximate solution found by these methods may serve as the starting value for the exact solution methods.

The exact solution methods are designed for the accurate computation of some or all the eigenvalues and the corresponding eigenvectors. These methods are divided into vector iteration methods (inverse iteration, forward iteration and simultaneous iteration methods), transformation methods (QR, Jacobi, Givens and Householder methods), Sturm sequence method and polynomial iteration method.

In vector iteration methods, both the eigenvalues and corresponding eigenvectors are found simultaneously, but in other exact solution methods, only eigenvalues are computed or the computed eigenvectors are, in general, not suitable for use in the final solutions. In such methods, another method such as the vector iteration method with a shift may be used for finding the eigenvector corresponding to a computed eigenvalue.

For a limited number of eigenvalues and corresponding eigenvectors of an eigenproblem for a large structure, the above solution methods have been modified or combined to take advantage of the useful characteristics of these methods.

The determinant search method (Bathe and Wilson, 1973; Gupta, 1973) combines the polynomial iteration method, the Sturm sequence method and the inverse iteration method. In this method, eigenvalues in a specified range are approximately isolated by using the bisection method and the Sturm sequence property and then located accurately by the polynomial iteration method. The corresponding eigenvectors are computed by inverse iteration with a shift. By this method, eigenvalues in any range and corresponding eigenvectors can be found. However, it has the disadvantage that the matrix is factorized in each iteration step to locate the eigenvalues of interest. This method is only efficient in the analysis of structures with small bandwidth.

The subspace iteration method (Bathe, 1971; Bathe and Wilson, 1972) is a combination of the simultaneous inverse iteration method and a Rayleigh-Ritz analysis. In this method, several independent vectors are improved by simultaneous inverse iteration step, and the best approximation to the eigenvectors is found in the subspace of the iteration vectors by a Rayleigh-Ritz analysis. In this method, eigenvalues at the end of the spectrum and the corresponding eigenvectors converge very rapidly.

Among the above eigenproblem solution methods, the subspace iteration method has hitherto been known to be very efficient for solving large eigenproblems, so this method has been widely used. However, the following shortcomings have been identified after extensive use of the method (Wilson and Itoh, 1983).

(1) When the number of eigenpairs to be required is large, the convergence of the required eigenvalues can be very slow.

(2) If a large number of eigenpairs are required, the computational effort required to form and solve the subspace eigenvalue problem can be significant.

(3) When the starting iteration vectors are poorly chosen, some of the eigenvalues and corresponding eigenvectors of interest may be missed.

To overcome the shortcomings of the subspace iteration method, many researchers have studied a variety of acceleration procedures of the subspace iteration method as follows.

Yamamoto and Ohtsubo (1976) used Chebyshev polynomials for acceleration. They showed that their method is more efficient than the basic subspace iteration method. Akl et al. (1979; 1982) employed over-relaxation method to accelerate the subspace iteration method and they have demonstrated the effectiveness of their method. Bathe and Ramaswamy (1980) used over-relaxation method and shifting techniques and they showed that the accelerated method could be applied effectively to the solution of eigenproblems in which the matrices have small or large bandwidths.

Nguyen and Arora (1980) developed the method for free vibration analysis of a large structure by partitioning it into a number of substructures to reduce the computer storage requirement. Cheu et al. (1987) investigated the effects of selecting initial vectors on computation efficiency for subspace iteration method.

Lam and Bertolini (1994; 1995) developed selective repeated inverse iteration and multiple inverse iteration for accelerated reduction of a subspace. Rajendran and Narasimhan (1994) used the another over-relaxation method. The method proposed by Bathe and Ramaswamy (1980) considers the acceleration of individual vectors using individual over-relaxation parameters, whereas the method proposed by Rajendran and Narasimhan considers the acceleration of the subspace as a whole. Qian and Dhatt (1995) developed to accelerate the subspace iteration by omitting some of the Rayleigh-Ritz procedure from certain iteration steps and obtaining a higher convergence rate.

Among the above accelerated techniques, a shifting technique is effectively used in the commercial FEM programs such as ADINA (1984). Since the singularity problem may occur during the use of the shifting technique in the accelerated scheme such as the subspace iteration method with shift, the shift must be within a limited region to avoid the singularity problem. In this study, a numerically stable method to remove the limitation in choosing a shift in the subspace iteration method with shift is developed.

1.2.2 Technique for Checking Missed Eigenvalues

The well-known Sturm sequence property has hitherto been applied to check the missed eigenvalues (Meirovitch, 1980; Hughes, 1987; Petyt, 1990; Bahte, 1996). The technique for checking the missed eigenvalues using the Sturm sequence property is very important in that it is the only means to make sure that indeed the required number of eigenvalues has been evaluated. As discussed in the background of this study, this checking process is required to obtain the exact dynamic response. the technique using the Sturm sequence property is used in the commercial FEM program such as ADINA. However, this technique can only be applied to the eigenproblem without the damping matrix such as the case of the undamped and proportional damped system (Newland, 1989).

In most real systems, the damping matrix is nonproportional (Caughey and O'Kelly, 1965). Even when a proportional damping is assumed for each sub-system in the analysis of soil-structure interaction problem, structural control problem, composite structures, etc., the resulting damping for the complete structure will be nonproportional. In these cases, the eigenvalue problem including the damping matrix should be analyzed for the exact dynamic response. By using the subspace iteration method (Olson and Vandini, 1989; Leung, 1995), the Lanczos method (Chen and Taylor, 1988; Rajakumar, 1993), the Arnoldi method (Arnoldi, 1951; Ren and Zheng, 1997) and the method developed by Lee and Kim (1998; 1999), one easily obtain the solution of this eigenproblem. However, although the eigenvalue analysis is performed well, the accuracy of the dynamic response is not guaranteed because the missed eigenvalues may exist in the required ones. To guarantee the exact dynamic response, a technique that can check the missed eigenvalues of structures with nonproportional damping (i.e., the eigenproblem including the damping matrix) is required.

In the field of the structural dynamics, the study on a technique for checking the missed eigenvalues in the eigenvalue analysis including the damping matrix has been

hardly carried out up to now. On the other hand, in the field of the control system engineering, the relevant research has been carried out a few (Locher, 1993; Tsai and Chen, 1993; Yamada et al., 1998). In the above literature, for stability test of a system having complex zeros, the root distribution of the characteristic polynomial of the system was investigated. However, the techniques presented in the above literature are improper to be applicable to an eigenproblem including damping matrix of a large system, because all the techniques calculate zeros of a polynomial by using rigorous symbolic algebraic operations.

As seen from the above review, a technique for checking the missed eigenvalues to be applicable to the eigenvalue analysis for the nonproportionally damped system does not developed yet. A technique for this case should be, therefore, developed to get the exact dynamic response as soon as possible.

1.3 Objectives and Scopes

The purpose of this study is to improve the subspace iteration method with shift that has been known to be very efficient for large eigenvalue problems and to develop a technique for checking the missed eigenvalues of structures with nonproportional damping.

First, the objectives and scopes of the study on an improved subspace iteration method with shift can be summarized as follows:

(1) Development of an improved subspace iteration method with shift that can obtain the desired eigenpairs without any singularity problem:

The side conditions are introduced to solve the singularity problem of the conventional subspace iteration method with shift. The nonsingularity of the proposed method is proved analytically. It is verified that the convergence of the proposed method is at least equal to that of the conventional subspace iteration method with shift, which is also proved analytically.

Next, the objectives and scopes of the study on a technique for checking the missed eigenvalues for the nonproportionally damped structure can be summarized as follows:

(1) Selection of an appropriate mathematical property to be applicable to a technique for checking the missed eigenvalues of structures with nonproportional damping:

An appropriate mathematical property is selected through a comprehensive review on various mathematical properties such as the extension of the Sturm sequence property, the Gerschgorin's theorem, the Routh-Hurwitz criterion and the argument principle.

(2) Development of a technique for checking the missed eigenvalues that can be applicable to the nonproportionally damped structures by using the appropriate mathematical property:

Since an analytical solution by symbolic operations cannot be calculated, the numerical solution by complex arithmetic operations is considered. And, by analyzing the numerical examples, it is verified that the proposed method can check exactly the missed eigenvalues and can be applicable to the large eigenproblem.

1.4 Organization

This dissertation consists of four chapters. Chapter 1 discusses the background, the literature review, and the objectives and scopes of this study.

In Chapter 2, a numerically stable eigenproblem solution method by improving the subspace iteration method is discussed. The conventional subspace iteration method with shift is reviewed in Section 2.1. In Section 2.2, an improved subspace iteration method using side conditions is proposed, and its numerical stability is proved analytically. The convergence analysis of the proposed method is performed and the comparison of operation counts of two methods, the conventional subspace iteration method with shift and the proposed method, is also described in this section. To show the effectiveness of the proposed method, two structures with distinct eigenvalues such as a three-dimensional frame structure and a simply supported rectangular plate structure and two structures with multiple eigenvalues such as a three-dimensional frame structure with a symmetric cross-section and a simply supported square plate structure are considered in Section 2.3.

In Chapter 3, a technique for checking the missed eigenvalues of structures with nonproportional damping is presented. The nonproportionally damped system is explained and various mathematical properties are reviewed in Section 3.1. Also, the argument principle chosen as the appropriate mathematical property through the comprehensive review on various properties is introduced in detail. In Section 3.2, a technique for checking the missed eigenvalues using the argument principle is presented. To verify the effectiveness of the proposed method, two numerical examples such as a simple spring-mass-damper system and a three-dimensional frame structure with concentrated dampers are considered in Section 3.3.

Finally, the conclusions of this study are summarized and some recommendations for the further study are also presented in Chapter 4.

CHAPTER 2

IMPROVED SUBSPACE ITERATION METHOD WITH SHIFT

2.1 Conventional Subspace Iteration Method with Shift

2.1.1 Algorithm

The general eigenvalue problem of the structural dynamics may be written as follows (Bathe, 1996):

$$\boldsymbol{K}\boldsymbol{X} = \boldsymbol{M}\boldsymbol{X}\boldsymbol{\Lambda} \tag{2.1}$$

where $\boldsymbol{K}$ and $\boldsymbol{M}$ are the stiffness matrix and the mass matrix of the discrete or discretized system of order n, respectively, the columns of $\boldsymbol{X}$ the eigenvectors, and $\boldsymbol{\Lambda}$ a diagonal matrix with eigenvalues.

Applying a shift μ to equation (2.1) gives

$$(\boldsymbol{K} - \mu\boldsymbol{M})\boldsymbol{X} = \boldsymbol{M}\boldsymbol{X}\boldsymbol{\Omega} \tag{2.2}$$

where

$$\boldsymbol{\Omega} = \boldsymbol{\Lambda} - \mu\boldsymbol{I} \tag{2.3}$$

and $\boldsymbol{I}$ is the identity matrix of order n.

Suppose that the p smallest eigenvalues $\lambda_i (i = 1,2,..., p)$ and corresponding eigenvectors $\boldsymbol{x}_i$ are required. Then the *j*th trial vector converges linearly to $\boldsymbol{x}_j$ at the rate of $(\lambda_j - \mu)/(\lambda_{p+1} - \mu)$. For faster convergence, q trial vectors are normally used with $q = min\{2p, p+8\}$. The details will be described in Section 2.1.3.

If we have p initial independent vectors $\boldsymbol{x}_i^{(0)}(i=1,2,...,p)$ spanning p-dimensional subspace in the neighborhood of the subspace of the desired eigenvectors and the approximate eigenvectors and corresponding eigenvalues after k iterations are denoted by $\boldsymbol{x}_i^{(k)}$ and $\lambda_i^{(k)}$, the conventional subspace iteration method with the shift μ for the (k+1)th iteration may be described as follows:

Step 1. Find the eigenvector approximations $\overline{\boldsymbol{X}}^{(k+1)} = \left[\overline{\boldsymbol{x}}_1^{(k+1)}, \overline{\boldsymbol{x}}_2^{(k+1)}, \cdots, \overline{\boldsymbol{x}}_q^{(k+1)}\right]$ by the simultaneous inverse iteration method:

$$(\boldsymbol{K} - \mu \boldsymbol{M})\overline{\boldsymbol{X}}^{(k+1)} = \boldsymbol{M}\boldsymbol{X}^{(k)} \tag{2.4}$$

where $\overline{\boldsymbol{X}}^{(k+1)}$ and $\boldsymbol{X}^{(k)}$ are the $(n \times q)$ matrices.

Step 2. Compute the projections of the matrices $(\boldsymbol{K} - \mu \boldsymbol{M})$ and $\boldsymbol{M}$ onto the subspace spanned by the q vectors in $\overline{\boldsymbol{X}}^{(k+1)}$:

$$\overline{\boldsymbol{K}}^{(k+1)} = \overline{\boldsymbol{X}}^{(k+1)^T}(\boldsymbol{K} - \mu \boldsymbol{M})\overline{\boldsymbol{X}}^{(k+1)} \text{ and} \tag{2.5}$$

$$\overline{\boldsymbol{M}}^{(k+1)} = \overline{\boldsymbol{X}}^{(k+1)^T}\boldsymbol{M}\overline{\boldsymbol{X}}^{(k+1)} \tag{2.6}$$

where $\overline{\boldsymbol{K}}^{(k+1)}$ and $\overline{\boldsymbol{M}}^{(k+1)}$ are the $(q \times q)$ symmetric matrices.

Step 3. Solve the eigenproblem of the reduced order q:

$$\overline{\boldsymbol{K}}^{(k+1)}\boldsymbol{Q}^{(k+1)} = \overline{\boldsymbol{M}}^{(k+1)}\boldsymbol{Q}^{(k+1)}\boldsymbol{\Omega}^{(k+1)} \tag{2.7}$$

where $\boldsymbol{Q}^{(k+1)}$ and $\boldsymbol{\Omega}^{(k+1)}$ are the $(q \times q)$ matrices.

Step 4. Find the improved eigenvectors $\boldsymbol{X}^{(k+1)}$ from the $(n \times q)$ matrix of Ritz trial vectors $\overline{\boldsymbol{X}}^{(k+1)}$, and the $(q \times q)$ projected system eigenvectors $\boldsymbol{Q}^{(k+1)}$:

$$\boldsymbol{X}^{(k+1)} = \overline{\boldsymbol{X}}^{(k+1)}\boldsymbol{Q}^{(k+1)}. \tag{2.8}$$

And the improved eigenvalues can be computed as follows:

$$\boldsymbol{\Lambda}^{(k+1)} = \boldsymbol{\Omega}^{(k+1)} + \mu\boldsymbol{I}. \tag{2.9}$$

Then, provided that the trial vectors in $\boldsymbol{X}^{(1)}$ are not orthogonal to one of the required eigenvectors and assuming an appropriate ordering of the trial vectors, $\boldsymbol{\Lambda}^{(k+1)}$ converges to $\boldsymbol{\Lambda}$ and $\boldsymbol{X}^{(k+1)}$ converges to $\boldsymbol{X}$ as *k* approaches infinity.

2.1.2 Condition on Shift

While the shifting procedure improves the convergence rate of the conventional subspace iteration method, it needs extra operations. The shifting technique will only be, therefore, performed when a criterion determines that the convergence will be improved sufficiently to cover the cost of the extra triangular factorization (Bathe and Ramaswamy, 1980).

If a shift is an eigenvalue itself or very close to it, all iteration vectors immediately converge to the eigenvector corresponding to that eigenvalue. The iteration vectors can then not be orthogonalized any more and the iteration procedure becomes unstable. If the shift is very close to an eigenvalue, the last pivot element in the $\boldsymbol{LDL}^{\boldsymbol{T}}$ factorization of the coefficient matrix usually becomes small compared with its original value and the coefficient matrix becomes close to singular.

To avoid this singularity, that is, to guarantee the numerical stability of the conventional subspace iteration method with shift, the following condition was adopted in the conventional subspace iteration method with shift (Bathe and Ramaswamy, 1980);

$$1.01\overline{\lambda}_{s-1} \le \mu \le 0.99\overline{\lambda}_{s} \tag{2.10}$$

where $\overline{\lambda}_{s-1}$ is the calculated approximation to the (*s*-1)th eigenvalue and $\overline{\lambda}_s$ the *s*th eigenvalue.

It means that a shift must be within a limited region resulting in slow convergence. Moreover, if the calculated approximation to an eigenvalue slightly differs from it, an eigenvalue may be inside the limited region. Then, the singularity may occur although a shift is inside the limited region. These are the significant disadvantages of the conventional subspace iteration method with shift. The purpose of this study is to remove the limitation in equation (2.10) for choosing the value of a shift μ.

2.1.3 Convergence Rate and Operation Count

With an adequate choice of the starting vectors, the conventional subspace iteration method with shift gives good approximations to the exact eigenvalues and eigenvectors even after only a few iterations. However, the subsequent convergence is only linear with the rates of convergence equal to $\frac{\lambda_i - \mu}{\lambda_{q+1} - \mu} \; (i = 1,2,..., p)$ for the *i*th eigenvector and $\left(\frac{\lambda_i - \mu}{\lambda_{q+1} - \mu} \right)^2$ for the corresponding eigenvalue (Bathe et al., 1977). These ratios indicate that for the higher eigenvalue convergence is slower. Hence, the convergence of the *p*th mode controls the termination of the iteration process.

One of the most important indicators of the effectiveness of numerical methods is the total number of operations required for finding a solution, which depends on both the rate of convergence and the number of operations per iteration.

Consider the number of Central Processor operations in order to obtain an estimate of the cost required for solving an eigenproblem. The actual cost must include, of course, the cost of the Peripheral Processor time. However, this time is not considered in this

investigation since it depends on the system and the programming technique. Let one operation equal to one multiplication that is nearly always followed by an addition. Assume that the half-bandwidths of $\boldsymbol{K}$ and $\boldsymbol{M}$ are m_K and m_M, respectively.

The total number of operations for the conventional subspace iteration method with shift, N_s (see Table 2.1) may be expressed by

$$N_s = T_s qn(2m_K + 4m_M + 2q + 4) + n(m_K^2 + 3m_K + 2m_M + 2) \tag{2.11}$$

where n and q are the number of equation of a system and the number of iteration vectors, respectively, and T_s is the total number of iterations required for the solution.

The steps for the conventional subspace iteration method with shift with the operations are summarized in Table 2.1.

Table 2.1 Operation count for the conventional subspace iteration method with shift

Operation	Calculation	Number of Operations
Multiplication	$\boldsymbol{K}-\mu\boldsymbol{M}$	$n(m_M+1)$
Factorization	$\boldsymbol{LDL}^T=\boldsymbol{K}-\mu\boldsymbol{M}$	$nm_M(m_M+3)/2$
Iteration		
Multiplication	$\boldsymbol{M}\boldsymbol{X}^{(k)}$	$qn(2m_M+1)$
Solve for $\overline{\boldsymbol{X}}^{(k+1)}$	$(\boldsymbol{K}-\mu\boldsymbol{M})\overline{\boldsymbol{X}}^{(k+1)}=\boldsymbol{M}\boldsymbol{X}^{(k)}$	$qn(2m_K+1)$
Multiplication	$\overline{\boldsymbol{K}}^{(k+1)}=\overline{\boldsymbol{X}}^{(k+1)^T}\boldsymbol{M}\boldsymbol{X}^{(k)}$	$qn(q+1)/2$
Multiplication	$\boldsymbol{M}\overline{\boldsymbol{X}}^{(k+1)}$	$qn(2m_M+1)$
Multiplication	$\overline{\boldsymbol{M}}^{(k+1)}=\overline{\boldsymbol{X}}^{(k+1)^T}\boldsymbol{M}\overline{\boldsymbol{X}}^{(k+1)}$	$qn(q+1)/2$
Solve for $\boldsymbol{Z}^{(k+1)}$ & $\boldsymbol{\Omega}^{(k+1)}$	$\overline{\boldsymbol{K}}^{(k+1)}\boldsymbol{Z}^{(k+1)}=\overline{\boldsymbol{M}}^{(k+1)}\boldsymbol{Z}^{(k+1)}\boldsymbol{\Omega}^{(k+1)}$	$O(q^3)$ neglected
Multiplication	$\boldsymbol{X}^{(k+1)}=\overline{\boldsymbol{X}}^{(k+1)}\boldsymbol{Z}^{(k+1)}$	nq^2
Sturm Sequence Check		
Multiplication	$\boldsymbol{K}-\lambda_p\boldsymbol{M}$	$n(m_M+1)$
Factorization	$\boldsymbol{LDL}^T=\boldsymbol{K}-\lambda_p\boldsymbol{M}$	$nm_M(m_M+3)/2$
Total	$T_s qn(2m_K+4m_M+2q+4)+n(m_K^2+3m_K+2m_M+2)$	

2.2 Proposed Method

2.2.1 Theory

Let us consider the simultaneous inverse iteration step in the conventional subspace iteration method with shift,

$$(\boldsymbol{K} - \mu\boldsymbol{M})\overline{\boldsymbol{X}}^{(k+1)} = \boldsymbol{M}\boldsymbol{X}^{(k)}. \tag{2.12}$$

Since if a shift is very close to an eigenvalue, the singularity problem occurs during the $\boldsymbol{LDL}^T$ factorization process of the coefficient matrix $(\boldsymbol{K} - \mu\boldsymbol{M})$ in equation (2.12). Then, the (k+1)th eigenvector approximations $\overline{\boldsymbol{X}}^{(k+1)}$ cannot be acquired, and so the iteration procedure cannot be performed any more. This is a significant disadvantage of the conventional subspace iteration method with shift.

In this study, to solve the above singularity problem the following procedures are proposed. First, let us consider a shift close to multiple eigenvalues. To simplify the notation in this discussion, assume that multiple eigenvalues close to the shift is the lowest eigenvalues and the multiplicity of the lowest eigenvalues is s, that is, $\lambda_1 = \lambda_2 = \cdots = \lambda_s$.

If the multiplicity s is equal to 1, then the shift is close to a distinct eigenvalue. In other words, the eigenproblem that has a shift close to a distinct eigenvalue can be considered as the special case of the eigenproblem that has a shift close to multiple eigenvalues. Hence, by using the same procedure as the case that the shift is close to the multiple eigenvalues, the solution of the case that has the shift close to a distinct eigenvalue can be found.

Then, the inverse iteration step on the multiple eigenvalues can be expressed as follows:

$$(\boldsymbol{K} - \mu \boldsymbol{M})\, \overline{\boldsymbol{X}}_s^{(k+1)} = \boldsymbol{M}\, \boldsymbol{X}_s^{(k)} \boldsymbol{D}_s^{(k+1)} \tag{2.13}$$

where the $(n \times s)$ matrices $\boldsymbol{X}_s^{(k)} = \left[\boldsymbol{x}_1^{(k)}, \boldsymbol{x}_2^{(k)}, ..., \boldsymbol{x}_s^{(k)}\right]$, $\overline{\boldsymbol{X}}_s^{(k+1)} = \left[\overline{\boldsymbol{x}}_1^{(k+1)}, \overline{\boldsymbol{x}}_2^{(k+1)}, ..., \overline{\boldsymbol{x}}_s^{(k+1)}\right]$, the $(s \times s)$ matrix $\boldsymbol{D}_s^{(k+1)} = diag(d_{11}^{(k+1)}, d_{22}^{(k+1)}, ..., d_{ss}^{(k+1)})$ and the scalar $d_{ii}^{(k+1)}$ controls the length of the vector $\overline{x}_i^{(k+1)}$.

Because there are only $(n \times s)$ equations with $((n+1) \times s)$ unknowns, $(n \times s)$ components of $\overline{\boldsymbol{X}}_s^{(k+1)}$ and s components of $d_{ii}^{(k+1)}$, in equation (2.13), s side conditions must be introduced for the solution of equation (2.13). These conditions are that the current vector set ($\boldsymbol{X}_s^{(k)}$) is orthogonal to the incremental vector set ($\Delta \boldsymbol{X}_s^{(k)}$) with respect to $\boldsymbol{M}$; that is,

$$\boldsymbol{X}_s^{(k)T} \boldsymbol{M} \Delta \boldsymbol{X}_s^{(k)} = \boldsymbol{0}. \tag{2.14}$$

Adding the mass orthonormality relation, $\boldsymbol{X}_s^{(k)T} \boldsymbol{M}\, \boldsymbol{X}_s^{(k)} = \boldsymbol{I}_s$, to the side conditions, equation (2.14), yields

$$\boldsymbol{X}_s^{(k)T} \boldsymbol{M}\, \overline{\boldsymbol{X}}_s^{(k+1)} = \boldsymbol{I}_s \tag{2.15}$$

where

$$\overline{\boldsymbol{X}}_s^{(k+1)} = \boldsymbol{X}_s^{(k)} + \Delta \boldsymbol{X}_s^{(k)}. \tag{2.16}$$

The inverse iteration step on the other eigenvalues make use of the existing equation, equation (2.12); that is,

$$(\boldsymbol{K} - \mu \boldsymbol{M})\, \overline{\boldsymbol{X}}_{q\text{-}s}^{(k+1)} = \boldsymbol{M}\, \boldsymbol{X}_{q\text{-}s}^{(k)} \tag{2.17}$$

where

$$\boldsymbol{X}_{q\text{-}s}^{(k)} = \left[\boldsymbol{x}_{s+1}^{(k)}, \boldsymbol{x}_{s+2}^{(k)}, \cdots, \boldsymbol{x}_{q}^{(k)}\right]. \tag{2.18}$$

Writing equations (2.13), (2.15) and (2.17) in matrix form gives

$$\begin{bmatrix} \boldsymbol{K} - \mu \boldsymbol{M} & \boldsymbol{M}\boldsymbol{X}_{s}^{(k)} \\ \boldsymbol{X}_{s}^{(k)^T} \boldsymbol{M} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \overline{\boldsymbol{X}}^{(k+1)} \\ \overline{\boldsymbol{D}}^{(k+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}\boldsymbol{X}^{(k)} \\ \boldsymbol{E} \end{bmatrix} \tag{2.19}$$

where the unknown $(s \times q)$ matrix $\overline{\boldsymbol{D}}^{(k+1)} = \left[\boldsymbol{D}_{s}^{(k+1)}, \boldsymbol{0}, ..., \boldsymbol{0}\right]$ and the $(s \times q)$ matrix $\boldsymbol{E} = \left[\boldsymbol{I}_{s}, \boldsymbol{0}, ..., \boldsymbol{0}\right]$.

Note that $\overline{\boldsymbol{X}}^{(k+1)}$ from equation (2.19) is used for $\overline{\boldsymbol{X}}^{(k+1)}$ in equations (2.5) and (2.6) instead of $\overline{\boldsymbol{X}}^{(k+1)}$ in equation (2.4). Equation (2.19) is the main linear algebraic equation used in the proposed method. The coefficient matrix of equation (2.19) is of order $(n + s)$, symmetric, and nonsingular. The nonsingularity is one of the significant advantages of the proposed method and will be shown in the next section.

The proposed method can be applied to practical problem as follows. After let us assume that a shift is very close to a distinct eigenvalue, one performs the factorizing process of the coefficient matrix with one side condition. If the shift is not very close to a distinct eigenvalue, but multiple ones (multiplicity = *s*), the $(n - s)$th pivot element in the factorizing process of the coefficient matrix usually becomes small compared with its original value and the coefficient matrix becomes singular. To avoid the singularity, the $(s - 1)$ side conditions are added, and then the factorizing process of the coefficient matrix is continued. Since the storage scheme of the proposed method uses the skyline algorithm, the extra operation number due to the $(s - 1)$ additional side conditions is a

few compared with the total operation number of the factorizing process of the coefficient matrix.

2.2.2 Proof of Nonsingularity of Coefficient Matrix

The most remarkable characteristic of the proposed method is that nonsingularity is always guaranteed. Let the coefficient matrix of equation (2.19) be denoted by $\boldsymbol{C}$, that is

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{K} - \mu \boldsymbol{M} & \boldsymbol{M}\boldsymbol{X}_s^{(k)} \\ \boldsymbol{X}_s^{(k)T}\boldsymbol{M} & \boldsymbol{0} \end{bmatrix}. \tag{2.20}$$

If $\boldsymbol{C}$ is nonsingular when the shift μ becomes multiple eigenvalues, that is, $\mu = \lambda_1 = \cdots = \lambda_s$, then it will be also nonsingular for a non-close shift. The resulting $\boldsymbol{C}^*$ will be

$$\boldsymbol{C}^* = \begin{bmatrix} \boldsymbol{K} - \lambda_s \boldsymbol{M} & \boldsymbol{M}\boldsymbol{X}_s \\ \boldsymbol{X}_s^T\boldsymbol{M} & \boldsymbol{0} \end{bmatrix}. \tag{2.21}$$

To show that the coefficient matrix $\boldsymbol{C}^*$ is always nonsingular, consider another matrix such as $\boldsymbol{Y}^T\boldsymbol{C}^*\boldsymbol{Y}$ where $\boldsymbol{Y}$ is a nonsingular square matrix of order $(n+s)$. The determinant property, $det\,\boldsymbol{Y}^T\boldsymbol{C}^*\boldsymbol{Y} = det\,\boldsymbol{Y}^T\,det\,\boldsymbol{C}^*\,det\,\boldsymbol{Y}$, provides $det\,\boldsymbol{Y}^T\boldsymbol{C}^*\boldsymbol{Y} \neq 0$ if and only if $det\,\boldsymbol{C}^* \neq 0$ and $det\,\boldsymbol{Y} \neq 0$. Hence, if it is proved that the determinant of $\boldsymbol{Y}^T\boldsymbol{C}^*\boldsymbol{Y}$ is non-zero, then the determinant of the coefficient matrix $\boldsymbol{C}^*$ may also be non-zero and $\boldsymbol{C}^*$ is nonsingular

In this paper, the matrix $\boldsymbol{Y}$ is assumed as

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{\Psi} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_s \end{bmatrix} \tag{2.22}$$

where $\boldsymbol{I}_s$ is an identity matrix of order s and $\boldsymbol{\Psi}$ is a set of arbitrary independent vectors containing the corresponding eigenvectors of multiple eigenvalues λ_s of the system as follows:

$$\boldsymbol{\Psi} = \left[\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \cdots, \boldsymbol{\psi}_{n\text{-}s}, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_s\right] \quad \text{when } \boldsymbol{X}_s = \left[\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_s\right] \tag{2.23}$$

where $\boldsymbol{\psi}_i$'s are arbitrary independent vectors chosen to be independent to the eigenvectors $\boldsymbol{x}_i$'s. Pre- and post-multiplying $\boldsymbol{Y}^T$ and $\boldsymbol{Y}$ to $\boldsymbol{C}^*$ yield

$$\boldsymbol{Y}^T\boldsymbol{C}^*\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{\Psi}^T(\boldsymbol{K} - \lambda_s\boldsymbol{M})\boldsymbol{\Psi} & \boldsymbol{\Psi}^T\boldsymbol{M}\boldsymbol{X}_s \\ \boldsymbol{X}_s^T\boldsymbol{M}\boldsymbol{\Psi} & \boldsymbol{0} \end{bmatrix}. \tag{2.24}$$

Considering the eigenvalue problem $(\boldsymbol{K} - \lambda_s\boldsymbol{M})\boldsymbol{X}_s = \boldsymbol{0}$ yields

$$\boldsymbol{\Psi}^T(\boldsymbol{K} - \lambda_s\boldsymbol{M})\boldsymbol{\Psi} = \begin{bmatrix} \tilde{\boldsymbol{A}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \tag{2.25}$$

where $\tilde{\boldsymbol{A}}$ is non-zero $(n-s)\times(n-s)$ submatrix.

The submatrix $\tilde{\boldsymbol{A}}$ is a nonsingular matrix, $det\ \tilde{\boldsymbol{A}} \neq 0$, having order of $(n-s)$ and rank of $(n-s)$, because it is given by eliminating the columns and rows having all zero elements from $\boldsymbol{\Psi}^T(\boldsymbol{K} - \lambda_s\boldsymbol{M})\boldsymbol{\Psi}$ of order n and rank $(n-s)$.

The orthonormal condition yields

$$\boldsymbol{\Psi}^T \boldsymbol{M} \boldsymbol{X}_s = \begin{bmatrix} \tilde{\boldsymbol{B}} \\ \boldsymbol{I}_s \end{bmatrix} \tag{2.26}$$

$$\boldsymbol{X}_s^T \boldsymbol{M} \boldsymbol{\Psi} = \begin{bmatrix} \tilde{\boldsymbol{B}} \\ \boldsymbol{I}_s \end{bmatrix}^T . \tag{2.27}$$

Substituting equations (2.26) and (2.27), into equation (2.25) yields

$$\boldsymbol{Y}^T \boldsymbol{C}^* \boldsymbol{Y} = \begin{bmatrix} \tilde{\boldsymbol{A}} & \boldsymbol{0} & \tilde{\boldsymbol{B}} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_s \\ \tilde{\boldsymbol{B}}^T & \boldsymbol{I}_s & \boldsymbol{0} \end{bmatrix}. \tag{2.28}$$

By applying the matrix determinant property of partitioned matrices (Franklin et al, 1994; Ogata, 1995), the determinant of equation (2.24) can be rewritten as

$$det\, \boldsymbol{Y}^T \boldsymbol{C}^* \boldsymbol{Y} = det \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_s \\ \boldsymbol{I}_s & \boldsymbol{0} \end{bmatrix} det \left(\tilde{\boldsymbol{A}} - \begin{bmatrix} \boldsymbol{0} & \tilde{\boldsymbol{B}} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_s \\ \boldsymbol{I}_s & \boldsymbol{0} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{0} \\ \tilde{\boldsymbol{B}}^T \end{bmatrix} \right) \tag{2.29}$$

or

$$det\, \boldsymbol{Y}^T \boldsymbol{C}^* \boldsymbol{Y} = det\, \tilde{\boldsymbol{A}} \neq 0 . \tag{2.30}$$

The determinant of $\boldsymbol{C}^*$, thus, is not equal to zero. The proof is completed mathematically for the nonsingularity of the coefficient matrix of the proposed method. That is, the numerical stability of the proposed method is proved analytically. The proposed method, therefore, has an advantage over the subspace iteration method with shift in that no limited regions are needed in the former.

2.2.3 Convergence Analysis

To analyze the convergence characteristics of the proposed method, we can use the concept of the convergence analysis of the conventional subspace iteration method (Bathe et al., 1977). The convergence of the subspace iterations is conveniently studied by first changing the basis from the finite element coordinate basis to the basis of the eigenvectors. This change of the basis is achieved using the following relation for the vectors $X^{(k)}$ in equation (2.13),

$$\boldsymbol{X}^{(k)} = \boldsymbol{\Phi}\boldsymbol{Z}^{(k)} \tag{2.31}$$

where $\boldsymbol{\Phi}$ is the matrix storing all eigenvectors, $\boldsymbol{\Phi} = \left[\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, ..., \boldsymbol{\phi}_n\right]$. Since $\boldsymbol{\Phi}$ is nonsingular, there is a unique $\boldsymbol{Z}^{(k)}$ for any $\boldsymbol{X}^{(k)}$, and vice versa.

Introducing the relation of equation (2.31) into equations (2.13) and (2.17), and premultiplying by $\boldsymbol{\Phi}^T$, we obtain the following equations;

$$(\boldsymbol{\Lambda} - \mu \boldsymbol{I}_n)\overline{\boldsymbol{Z}}_s^{(k+1)} = \boldsymbol{Z}_s^{(k)}\boldsymbol{D}_s^{(k+1)}, \text{ and} \tag{2.32}$$

$$(\boldsymbol{\Lambda} - \mu \boldsymbol{I}_n)\overline{\boldsymbol{Z}}_{q\text{-}s}^{(k+1)} = \boldsymbol{Z}_{q\text{-}s}^{(k)} \tag{2.33}$$

where the $(n \times s)$ matrix $\overline{\boldsymbol{Z}}_q^{(k+1)} = \left[\bar{\boldsymbol{z}}_1^{(k+1)}, \bar{\boldsymbol{z}}_2^{(k+1)}, ..., \bar{\boldsymbol{z}}_s^{(k+1)}\right]$ and the $(n \times (q-s))$ matrix $\overline{\boldsymbol{Z}}_{q\text{-}s}^{(k+1)} = \left[\bar{\boldsymbol{z}}_{s+1}^{(k+1)}, \bar{\boldsymbol{z}}_{s+2}^{(k+1)}, ..., \bar{\boldsymbol{z}}_q^{(k+1)}\right]$. And then, the equations equivalent to equations (2.5) to (2.8), but which express the relations in the new basis, are used to evaluate $\boldsymbol{Z}^{(k+1)}$. The convergence rate of the iteration is established from equations (2.32) and (2.33), and using the fact that in the subspace iterations always the optimum approximations to the required eigenvalues and eigenvectors are calculated.

For the convergence analysis let the iteration matrix $\boldsymbol{Z}^{(k)}$ be denoted as follows:

$$\boldsymbol{Z}^{(k)} = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & & 0 \\ 0 & & 1 & 0 & & 0 \\ 0 & & 0 & 1 & & 0 \\ 0 & & 0 & 0 & & 0 \\ \vdots & & \vdots & \vdots & & 0 \\ 0 & & 0 & 0 & & 1 \\ z^{(k)}_{q+1,1} & \cdots & z^{(k)}_{q+1,s} & z^{(k)}_{q+1,s+1} & \cdots & z^{(k)}_{q+1,q} \\ z^{(k)}_{q+2,1} & \cdots & z^{(k)}_{q+2,s} & z^{(k)}_{q+2,s+1} & \cdots & z^{(k)}_{q+2,q} \\ \vdots & & \vdots & \vdots & & \vdots \\ z^{(k)}_{n,1} & \cdots & z^{(k)}_{n,s} & z^{(k)}_{n,s+1} & \cdots & z^{(k)}_{n,q} \end{bmatrix} \tag{2.34}$$

where $\boldsymbol{Z}^{(k)}$ is completely general, because the unit $(q \times q)$ matrix $\boldsymbol{I}$ can always be obtained by linearly combining columns, provided $\boldsymbol{Z}^{(k)}$ is not deficient in the vectors $\boldsymbol{e}_j\,(j = 1,2,...,q)$, which are vectors of the order *n* with only zero elements except for the *j*th location which is unity. Using equations (2.32) and (2.33), we then obtain

$$\overline{\boldsymbol{Z}}^{(k+1)} = \begin{bmatrix} \frac{d_{11}^{(k+1)}}{\lambda_1-\mu} & \cdots & 0 & 0 & \cdots & 0 \\ 0 & & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & & 0 \\ 0 & & \frac{d_{ss}^{(k+1)}}{\lambda_s-\mu} & 0 & & 0 \\ 0 & & 0 & \frac{1}{\lambda_{s+1}-\mu} & & 0 \\ 0 & & 0 & 0 & & 0 \\ \vdots & & \vdots & \vdots & & 0 \\ 0 & & 0 & 0 & & \frac{1}{\lambda_q-\mu} \\ z_{q+1,1}^{(k)}\frac{d_{11}^{(k+1)}}{\lambda_{q+1}-\mu} & \cdots & z_{q+1,s}^{(k)}\frac{d_{ss}^{(k+1)}}{\lambda_{q+1}-\mu} & z_{q+1,s+1}^{(k)}\frac{1}{\lambda_{q+1}-\mu} & \cdots & z_{q+1,q}^{(k)}\frac{1}{\lambda_{q+1}-\mu} \\ z_{q+2,1}^{(k)}\frac{d_{11}^{(k+1)}}{\lambda_{q+2}-\mu} & \cdots & z_{q+2,s}^{(k)}\frac{d_{ss}^{(k+1)}}{\lambda_{q+2}-\mu} & z_{q+2,s+1}^{(k)}\frac{1}{\lambda_{q+2}-\mu} & \cdots & z_{q+2,q}^{(k)}\frac{1}{\lambda_{q+2}-\mu} \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{n,1}^{(k)}\frac{d_{11}^{(k+1)}}{\lambda_n-\mu} & \cdots & z_{n,s}^{(k)}\frac{d_{ss}^{(k+1)}}{\lambda_n-\mu} & z_{n,s+1}^{(k)}\frac{1}{\lambda_n-\mu} & \cdots & z_{n,q}^{(k)}\frac{1}{\lambda_n-\mu} \end{bmatrix}. \quad (2.35)$$

The subspace $\boldsymbol{E}^{(k+1)}$ spanned by $\overline{\boldsymbol{Z}}^{(k+1)}$ is not changed if we multiply the first *s* columns in $\overline{\boldsymbol{Z}}^{(k+1)}$ by $(\lambda_i-\mu)/d_{ii}^{(k+1)}$ and the other columns by $(\lambda_j-\mu)$, i.e., $\boldsymbol{E}^{(k+1)}$ is also spanned by $\widetilde{\boldsymbol{Z}}^{(k+1)}$, where

$$\widetilde{\boldsymbol{Z}}^{(k+1)} = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & & 0 \\ 0 & & 1 & 0 & & 0 \\ 0 & & 0 & 1 & & 0 \\ 0 & & 0 & 0 & & 0 \\ \vdots & & \vdots & \vdots & & 0 \\ 0 & & 0 & 0 & & 1 \\ z_{q+1,1}^{(k)} \dfrac{\lambda_1-\mu}{\lambda_{q+1}-\mu} & \cdots & z_{q+1,s}^{(k)} \dfrac{\lambda_s-\mu}{\lambda_{q+1}-\mu} & z_{q+1,s+1}^{(k)} \dfrac{\lambda_{s+1}-\mu}{\lambda_{q+1}-\mu} & \cdots & z_{q+1,q}^{(k)} \dfrac{\lambda_q-\mu}{\lambda_{q+1}-\mu} \\ z_{q+2,1}^{(k)} \dfrac{\lambda_1-\mu}{\lambda_{q+2}-\mu} & \cdots & z_{q+1,s}^{(k)} \dfrac{\lambda_s-\mu}{\lambda_{q+2}-\mu} & z_{q+2,s+1}^{(k)} \dfrac{\lambda_{s+1}-\mu}{\lambda_{q+2}-\mu} & \cdots & z_{q+1,q}^{(k)} \dfrac{\lambda_q-\mu}{\lambda_{q+2}-\mu} \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{n,1}^{(k)} \dfrac{\lambda_1-\mu}{\lambda_n-\mu} & \cdots & z_{q+1,s}^{(k)} \dfrac{\lambda_s-\mu}{\lambda_n-\mu} & z_{n,s+1}^{(k)} \dfrac{\lambda_{s+1}-\mu}{\lambda_n-\mu} & \cdots & z_{n,q}^{(k)} \dfrac{\lambda_q-\mu}{\lambda_n-\mu} \end{bmatrix}. \qquad (2.36)$$

In the subspace iteration using equations (2.32) and (2.33), the best eigenvector approximations are extracted from the vectors stored in $\overline{\boldsymbol{Z}}^{(k+1)}$. But on inspecting the columns of $\widetilde{\boldsymbol{Z}}^{(k+1)}$ in equation (2.36) we find that ultimately the *j*th column is the best approximation to the vector $\boldsymbol{e}_j$ in the subspace $\boldsymbol{E}^{(k+1)}$. The ultimate rate of convergence to the *j*th eigenvector is thus obtained by evaluating

$$\frac{\left\|\widetilde{\boldsymbol{z}}_j^{(k+1)} - \boldsymbol{e}_j\right\|_2}{\left\|\boldsymbol{z}_j^{(k)} - \boldsymbol{e}_j\right\|_2} = \frac{\lambda_j - \mu}{\lambda_{q+1} - \mu} \sqrt{\frac{\sum_{i=q+1}^{n} (z_{i,j}^{(k)})^2 \left(\dfrac{\lambda_{q+1}-\mu}{\lambda_i-\mu}\right)^2}{\sum_{i=q+1}^{n} (z_{i,j}^{(k)})^2}} \qquad (2.37)$$

where $\boldsymbol{z}_j^{(k)}$ is the *j*th column of $\boldsymbol{Z}^{(k)}$, and similar for $\widetilde{\boldsymbol{z}}_j^{(k+1)}$. Hence,

$$\frac{\left\|\tilde{\boldsymbol{z}}_j^{(k+1)} - \boldsymbol{e}_j\right\|_2}{\left\|\boldsymbol{z}_j^{(k)} - \boldsymbol{e}_j\right\|_2} \le \frac{\lambda_j - \mu}{\lambda_{q+1} - \mu} \tag{2.38}$$

and convergence is linear with the rate of convergence equal to $\frac{\lambda_j - \mu}{\lambda_{q+1} - \mu}$. We, therefore, conclude that provided the starting subspace is not orthogonal to the required least dominant subspace spanned by $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, ..., \boldsymbol{\phi}_q$, the *j*th column in $\boldsymbol{X}^{(k+1)}$ converges linearly with the rate $\frac{\lambda_j - \mu}{\lambda_{q+1} - \mu}$ to $\boldsymbol{\phi}_j$. Since the eigenvalues are calculated using the Rayleigh quotient, the *j*th eigenvalue in equation (2.7) converges linearly with the rate $\left\{\frac{\lambda_j - \mu}{\lambda_{q+1} - \mu}\right\}^2$ to λ_j.

2.2.4 Operation Count

As seen from Section 2.1.3, let one operation equal to one multiplication that is nearly always followed by an addition. Assume that the half-bandwidths of $\boldsymbol{K}$ and $\boldsymbol{M}$ are m_K and m_M, respectively.

The total number of operations for the proposed method, N_p (see Table 2.2), may be expressed by

$$\begin{aligned} N_p = T_p\{&qn(2m_K + 4m_M + 2q + 4 + s) \\ &+ sn(m_K + (s+1)/2)\} + n(m_K^2 + 3m_K + 2m_M + 2) \end{aligned} \tag{2.39}$$

where n and q are the number of equation of a system and the number of iteration vectors, respectively, s is the multiplicity of the multiple eigenvalues which is on or very close to a shift and T_p is the number of iterations required for the solution.

The steps for the proposed method with the operations are summarized in Table 2.2.

The proposed method needs more operations per each iteration step, $sn\{q + m_K + (s+1)/2\}$, than the conventional subspace iteration method with shift. Assume that the ratio is composed of the operation count per iteration of the proposed method (n_p), that of the conventional subspace iteration method with shift (n_s), and the difference of the operation count per iteration for above two methods ($n_p - n_s$) as follows:

$$ratio = \frac{n_p - n_s}{n_p} = \frac{sn\{q + m_K + (s+1)/2\}}{qn\{(2m_K + 4m_M + 2q + 4 + s) + sn(m_K + (s+1)/2)\}}. \quad (2.40)$$

Then, if the half-bandwidth of the stiffness matrix (m_K) is equal to that of the mass matrix (m_M), the above ratio can be approximated as follows:

$$ratio \approx \frac{s}{6q}. \quad (2.41)$$

This ratio means that the larger the number of the required eigenpairs, the smaller is the difference of the operation count between the proposed method and the conventional subspace iteration method with shift. That is, the number of operations for the aforementioned two methods, the conventional subspace iteration method with shift and the proposed method, is almost the same when the number of eigenpairs to be required is large.

Table 2.2 Operation count for the proposed method

Operation	Calculation	Number of Operations
Iteration		
k=0		
Multiplication	$\boldsymbol{K}-\mu\boldsymbol{M}$	$n(m_M+1)$
Multiplication	$\boldsymbol{M}\boldsymbol{X}^{(0)}$	$qn(2m_M+1)$
Change the last s columns of $\boldsymbol{K}-\mu\boldsymbol{M}$ into $\boldsymbol{M}\boldsymbol{X}_s^{(0)}$		neglected
Factorization	$\boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T=\boldsymbol{F}^{(1)}$	$\{nm_K(m_K+3+2s)+(s+1)\}/2$
k=1,2,…		
Multiplication	$\boldsymbol{M}\boldsymbol{X}^{(k)}$	$qn(2m_M+1)$
Change the last s columns of $\boldsymbol{K}-\mu\boldsymbol{M}$ into $\boldsymbol{M}\boldsymbol{X}_s^{(k)}$		neglected
Factorization	$\boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T=\boldsymbol{F}^{(k+1)}$	$sn\{m_K+(s+1)/2\}$
Solve for $\hat{\boldsymbol{X}}^{(k+1)}$	$\boldsymbol{F}^{(k+1)}\hat{\boldsymbol{X}}^{(k+1)}=\boldsymbol{R}$	$qn(2m_K+s+1)$
Multiplication	$\overline{\boldsymbol{K}}^{(k+1)}=\overline{\boldsymbol{X}}^{(k+1)^T}\boldsymbol{M}\boldsymbol{X}^{(k)}$	$qn(q+1)/2$
Multiplication	$\boldsymbol{M}\overline{\boldsymbol{X}}^{(k+1)}$	$qn(2m_M+1)$
Multiplication	$\overline{\boldsymbol{M}}^{(k+1)}=\overline{\boldsymbol{X}}^{(k+1)^T}\boldsymbol{M}\overline{\boldsymbol{X}}^{(k+1)}$	$qn(q+1)/2$
Solve for $\boldsymbol{Z}^{(k+1)}$ & $\boldsymbol{\Omega}^{(k+1)}$	$\overline{\boldsymbol{K}}^{(k+1)}\boldsymbol{Z}^{(k+1)}=\overline{\boldsymbol{M}}^{(k+1)}\boldsymbol{Z}^{(k+1)}\boldsymbol{\Omega}^{(k+1)}$	$O(q^3)$ neglected
Multiplication	$\boldsymbol{X}^{(k+1)}=\overline{\boldsymbol{X}}^{(k+1)}\boldsymbol{Z}^{(k+1)}$	nq^2
Sturm Sequence Check		
Multiplication	$\boldsymbol{K}-\lambda_p\boldsymbol{M}$	$n(m_M+1)$
Factorization	$\boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T=\boldsymbol{K}-\lambda_p\boldsymbol{M}$	$nm_K(m_K+3)/2$
Total	$T_p\{qn(2m_K+4m_M+2q+4+s)+sn(m_K+(s+1)/2)\}$ $+n(m_K^2+3m_K+2m_M+2)$	

where $\boldsymbol{F}^{(k+1)}=\begin{bmatrix}\boldsymbol{K}-\mu\boldsymbol{M} & \boldsymbol{M}\boldsymbol{X}_s^{(k)}\\ \boldsymbol{X}_s^{(k)^T}\boldsymbol{M} & 0\end{bmatrix}$, $\hat{\boldsymbol{X}}^{(k+1)}=\begin{bmatrix}\overline{\boldsymbol{X}}^{(k+1)}\\ \overline{\boldsymbol{D}}^{(k+1)}\end{bmatrix}$, $\boldsymbol{R}=\begin{bmatrix}\boldsymbol{M}\boldsymbol{X}^{(k)}\\ \boldsymbol{E}\end{bmatrix}$ (equation (2.19))

2.3 Numerical Examples

To verify the effectiveness of the proposed method, the following numerical examples are analyzed. First, the structures with the distinct eigenvalues are taken as examples. The three-dimensional frame structure used by Bathe and Wilson (1972) and the simply supported rectangular plate structure are analyzed. And the structures with multiple eigenvalues such as the three-dimensional frame structure with symmetric cross-section and the simply supported square plate structure are considered.

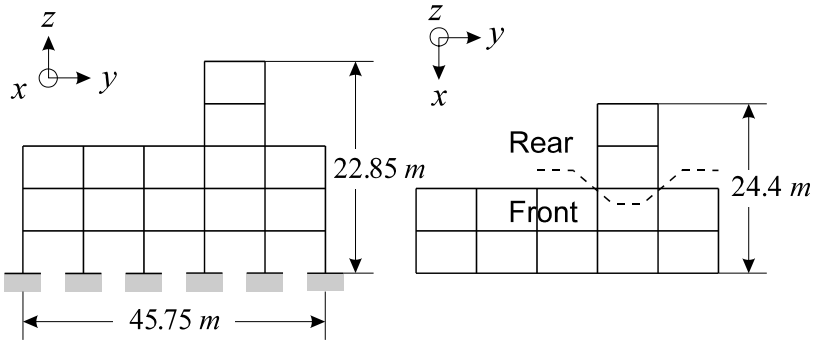
The solution time spent for the first ten eigenpairs and the convergence rate of the proposed method is compared with that of the conventional subspace iteration method with shift which is not used the limited region (see equation (2.10)). The predetermined physical error norm of 10^{-6}, which yields a numerically stable eigenproblem solution and sufficient accuracy in the calculated eigenvalues and eigenvectors for practical analysis (Bathe and Ramaswamy, 1980). The physical error norm (Bathe, 1996) is defined as

$$\text{physical error norm} = \frac{\left\| (\boldsymbol{K} - \lambda_i^{(k)} \boldsymbol{M}) \boldsymbol{x}_i^{(k)} \right\|_2}{\left\| \boldsymbol{K} \boldsymbol{x}_i^{(k)} \right\|_2}. \tag{2.42}$$

All executions are computed on the IRIS4D-20-S17 with 10 MIPS and 0.9 MFLOPS.

2.3.1 Three-Dimensional Frame Structure

The first example is a three-dimensional frame structure. Figure 2.1 shows the geometric configuration and material properties. The structure discretized by using 100 beam elements, resulting in system of dynamic equations with a total of 468 degrees of freedom. The consistent mass matrix is used for $\boldsymbol{M}$. The lowest ten eigenvalues of the three-dimensional frame structure are shown in Table 2.3. The lowest ten eigenvalues of this structure are well separated.



(a) Elevation (b) Plan

Column in Front Building : $A = 0.2787\ m^2, I = 8.631\times10^{-3}\ m^4$

Column in Rear Building : $A = 0.3716\ m^2, I = 10.787\times10^{-3}\ m^4$

All Beams into x - Direction : $A = 0.6096\ m^2, I = 6.473\times10^{-1}\ m^4$

All Beams into y - Direction : $A = 0.2787\ m^2, I = 8.631\times10^{-1}\ m^4$

$$E = 2.068\times10^{10}\ Pa,\ \rho = 5.154\times10^{2}\ kg/m^3$$

Figure 2.1 Three-dimensional frame structure

Some results are shown in Table 2.4 and in Figures 2.2 to 2.7. The solution time for two methods are summarized in Table 2.4. When a shift is on $1.01\lambda_5$, the subspace iteration method with shift and the proposed method obtain the required ten eigenpairs. However, when the shift is $1.00001\lambda_5$ or on λ_5, the subspace iteration method with shift does not calculate the solutions while the proposed method finds the solutions. It shows that the iteration procedure for the proposed method can converge without any singularity even if the shift is exactly the same as an eigenvalue, as analytically proved in Section 2.2.2.

For each solution method, the convergence of each eigenpair is depicted in Figures 2.2 to 2.7. Figures 2.2 and 2.3 show that when the shift is on $1.01\lambda_5$ the convergence of the proposed method is nearly equal to that of the subspace iteration method with shift. Figures 2.4 and 2.5 show that when the shift is on $1.00001\lambda_5$ the proposed method converges well without any singularity while the subspace iteration method with shift can not converge due to the singularity. Figures 2.6 to 2.7 show that show that when the shift is exactly the same as the fifth eigenvalue the proposed method only converges well without any singularity. As the above results, the proposed method can choose a more aggressive shift than the subspace iteration method with shift.

Table 2.3 The lowest ten eigenvalues of the three-dimensional frame structure

Mode number	Eigenvalues
1	0.4239E+00
2	0.5504E+00
3	0.8385E+00
4	0.1070E+01
5	0.1600E+01
6	0.2147E+01
7	0.2527E+01
8	0.3302E+01
9	0.3755E+01
10	0.4372E+01

Table 2.4 Solution time for the lowest ten eigenvalues of the three-dimensional frame structure

Analysis methods	Shift = 1.01 λ_5	Shift = 1.00001 λ_5	Shift = λ_5
Subspace iteration method with shift	485.36 (1.00)	No solution	No solution
Proposed method	492.16 (1.01)	492.02	491.70

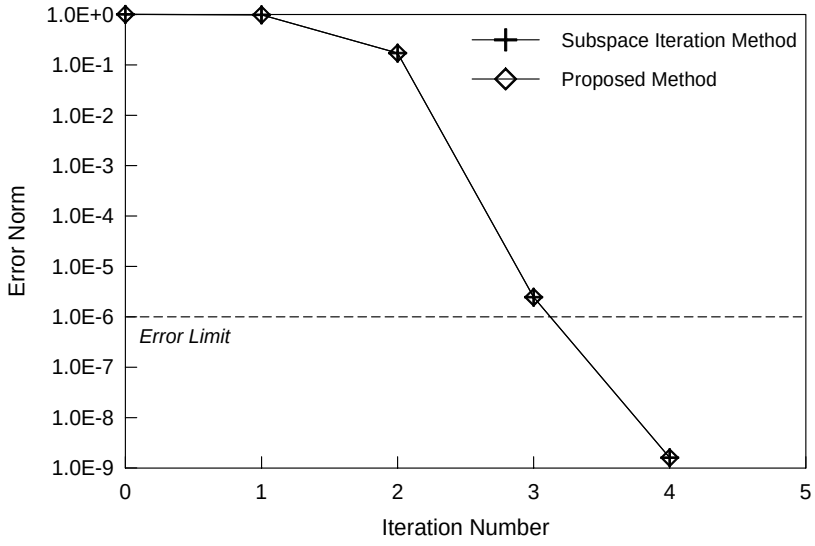


Figure 2.2 Error norm versus iteration number of the fifth eigenpair of the three-dimensional framed structure in the case of shift = $1.01\lambda_5$

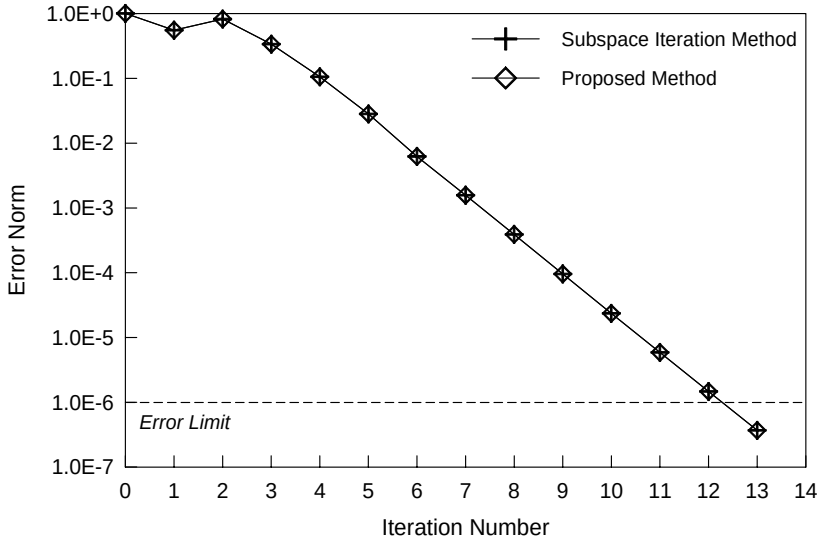


Figure 2.3 Error norm versus iteration number of the tenth eigenpair of the three-dimensional framed structure in the case of shift = $1.01\lambda_5$

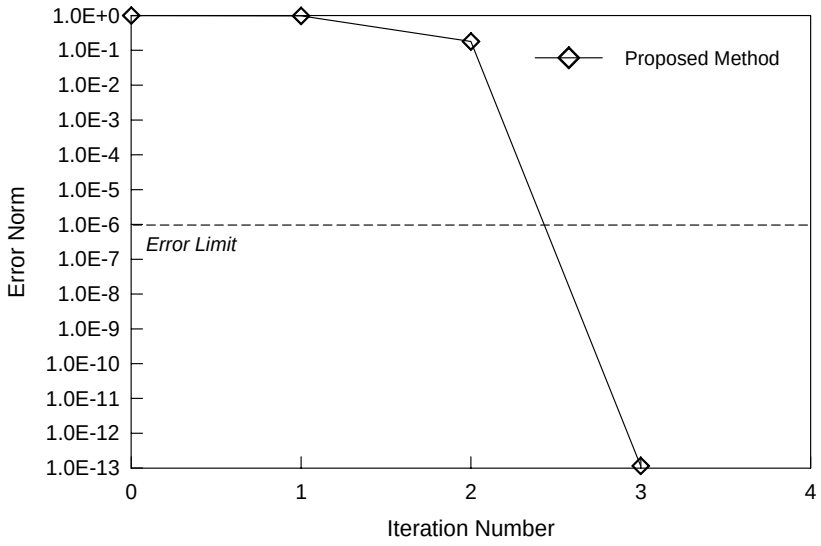


Figure 2.4 Error norm versus iteration number of the fifth eigenpair of the three-dimensional framed structure in the case of shift = 1.00001 λ_5

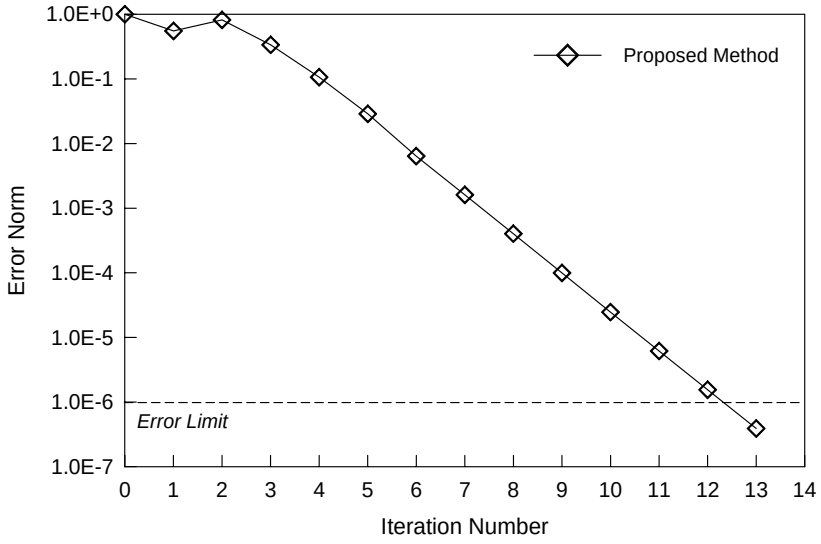


Figure 2.5 Error norm versus iteration number of the tenth eigenpair of the three-dimensional framed structure in the case of shift = 1.00001 λ_5

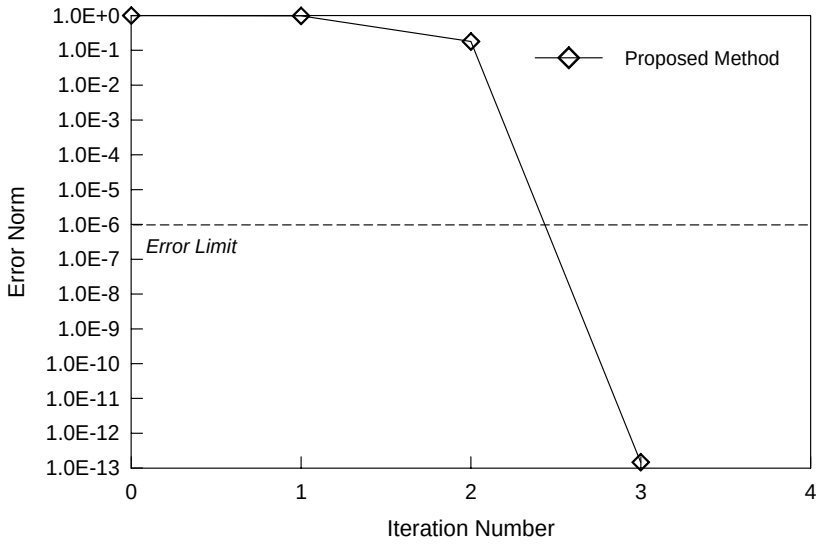


Figure 2.6 Error norm versus iteration number of the fifth eigenpair of the three-dimensional framed structure in the case of shift = λ_5

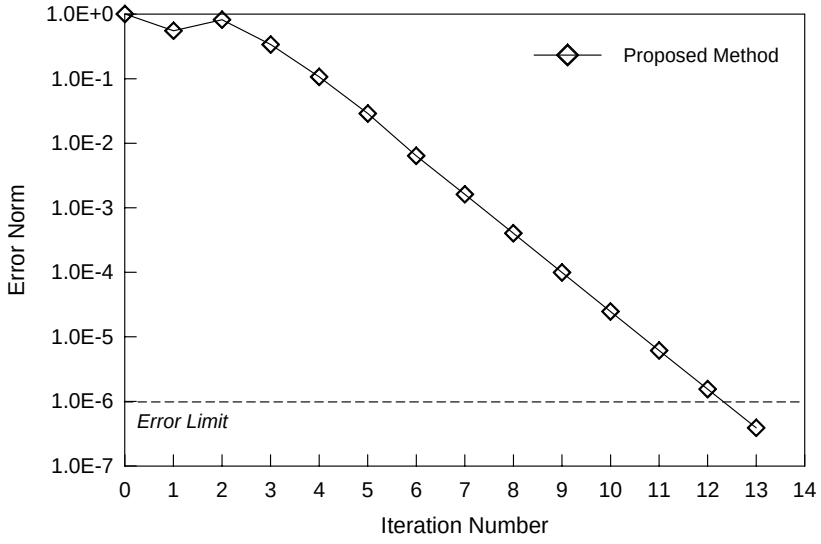
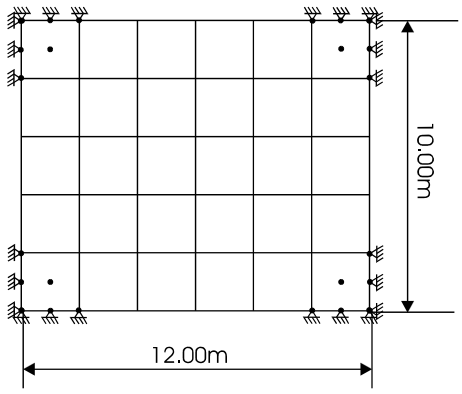


Figure 2.7 Error norm versus iteration number of the tenth eigenpair of the three-dimensional framed structure in the case of shift = λ_5

2.3.2 Simply Supported Rectangular Plate Structure

The second example is a simply supported rectangular plate structure. Figure 2.8 shows the geometric configuration and material properties. The structure discretized by using 30 shell elements (nine node per element), resulting in system of dynamic equations with a total of 583 degrees of freedom. The consistent mass matrix is used for $\boldsymbol{M}$. The lowest ten eigenvalues of the simply supported rectangular plate structure are shown in Table 2.5. All the calculated eigenvalues of this structure are distinct.



$E=2.0\times10^{11}Pa$, $\rho=7.85\times10^{3}kg/m^{3}$

Poisson ratio $=0.3$, *Shell thickness* $=0.01m$

Figure 2.8 Simply supported rectangular plate structure

Some results are shown in Table 2.6 and in Figures 2.9 to 2.14. The solution time for two methods are summarized in Table 2.6. When a shift is on $1.01\lambda_5$, the subspace iteration method with shift and the proposed method obtain the required ten eigenpairs. However, when the shift is on $1.000001\lambda_5$ or on λ_5, the subspace iteration method with

shift does not calculate the solutions while the proposed method finds the solutions. It shows that the iteration procedure for the proposed method can converge without any singularity even if the shift is exactly the same as an eigenvalue.

For each solution method, the convergence of each eigenpair is depicted in Figures 2.9 and 2.14. Figures 2.9 and 2.10 show that when the shift is on $1.01\lambda_5$ the convergence of the proposed method is nearly equal to that of the subspace iteration method with shift. Figures 2.11 and 2.12 show that when the shift is on $1.000001\lambda_5$ the proposed method converges well without any singularity while the subspace iteration method with shift can not converge due to the singularity. Figures 2.13 and 2.14 show that show that when the shift is exactly the same as the fifth eigenvalue the proposed method only converges well without any singularity.

Table 2.5 The lowest ten eigenvalues of the simply supported rectangular plate structure

Mode number	Eigenvalues
1	0.6628E+01
2	0.3423E+02
3	0.5534E+02
4	0.1116E+03
5	0.1405E+03
6	0.2666E+03
7	0.2786E+03
8	0.3849E+03
9	0.4592E+03
10	0.6248E+03

Table 2.6 Solution time for the lowest ten eigenvalues of the simply supported rectangular plate structure

Analysis methods	Shift = $1.01\lambda_5$	Shift = $1.000001\lambda_5$	Shift = λ_5
Subspace iteration method with shift	626.09 (1.00)	No solution	No solution
Proposed method	641.25 (1.02)	641.05	641.02

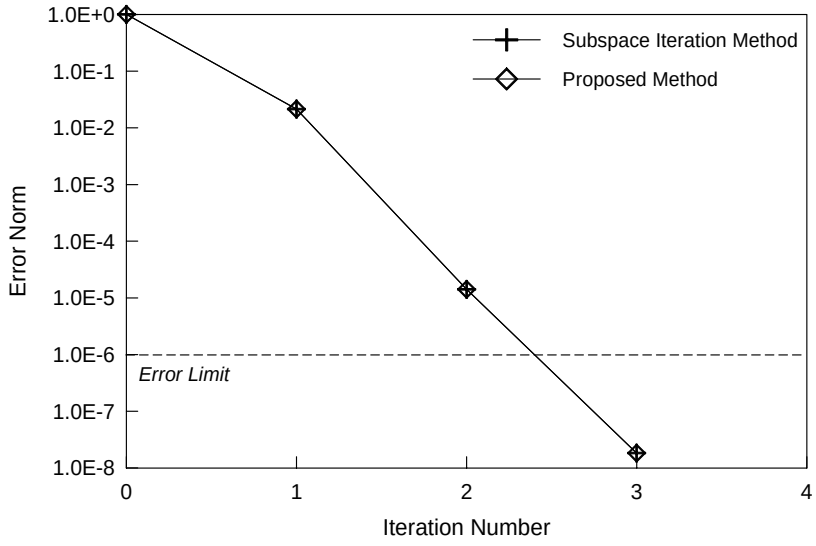


Figure 2.9 Error norm versus iteration number of the fifth eigenpair of the simply supported plate structure in the case of shift = 1.01 λ_5

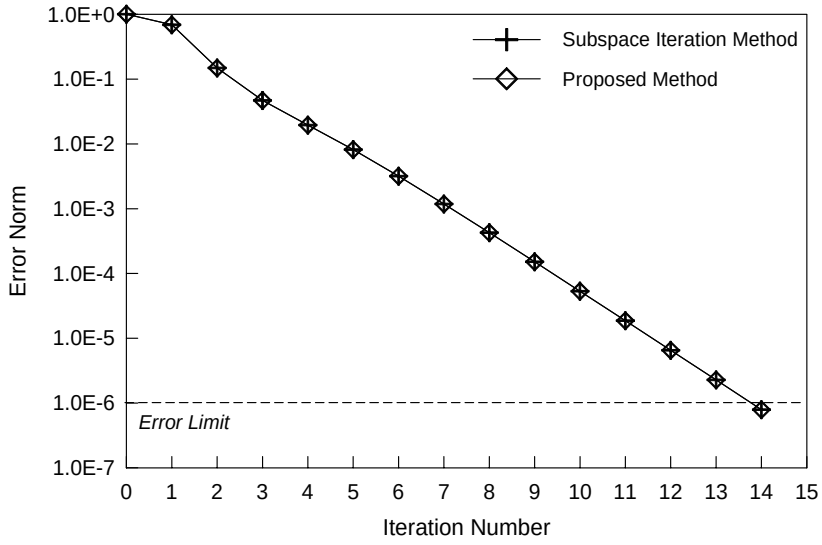


Figure 2.10 Error norm versus iteration number of the tenth eigenpair of the simply supported plate structure in the case of shift = 1.01 λ_5

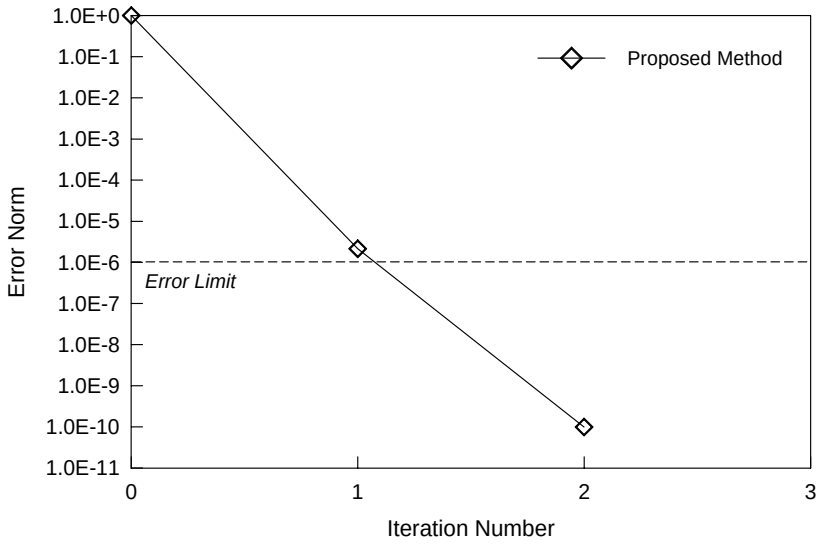


Figure 2.11 Error norm versus iteration number of the fifth eigenpair of the simply supported plate structure in the case of shift = 1.000001 λ_5

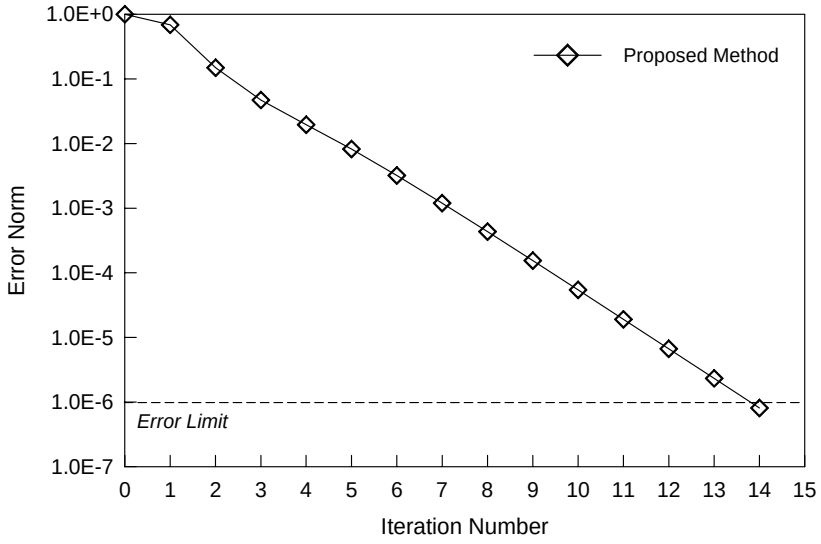


Figure 2.12 Error norm versus iteration number of the tenth eigenpair of the simply supported plate structure in the case of shift = 1.000001 λ_5

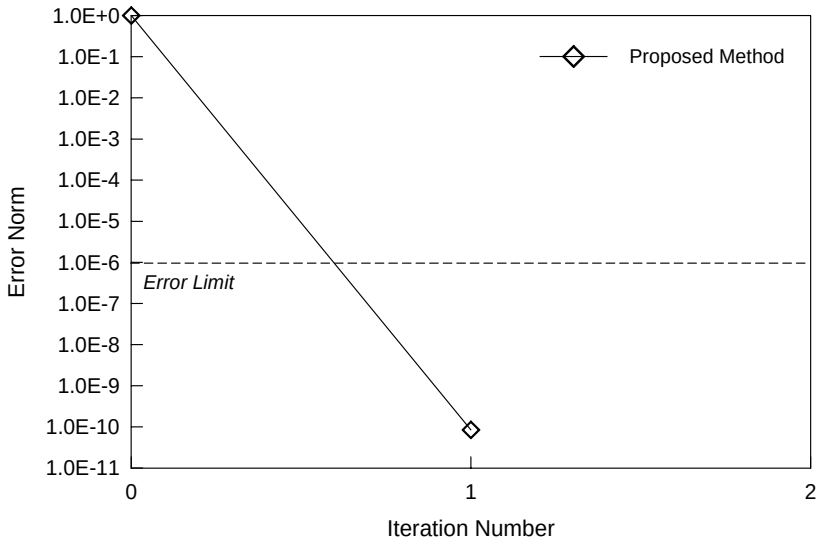


Figure 2.13 Error norm versus iteration number of the fifth eigenpair of the simply supported plate structure in the case of shift = λ_5

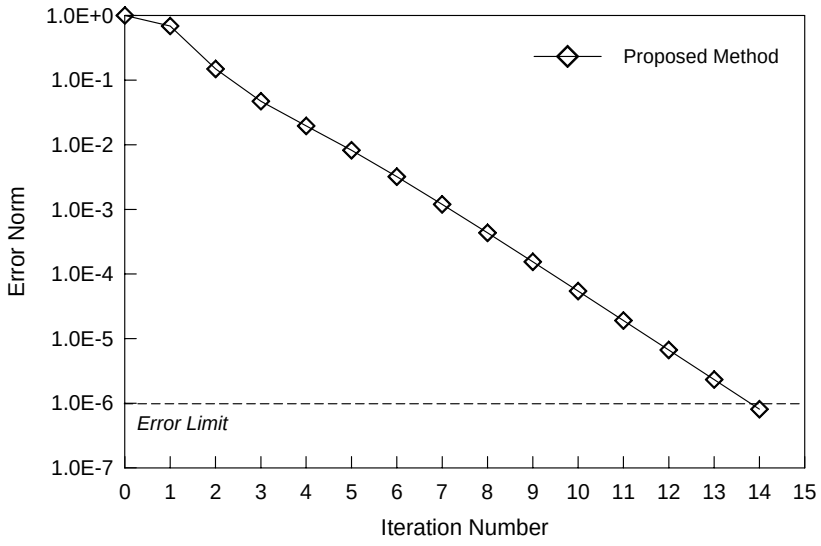
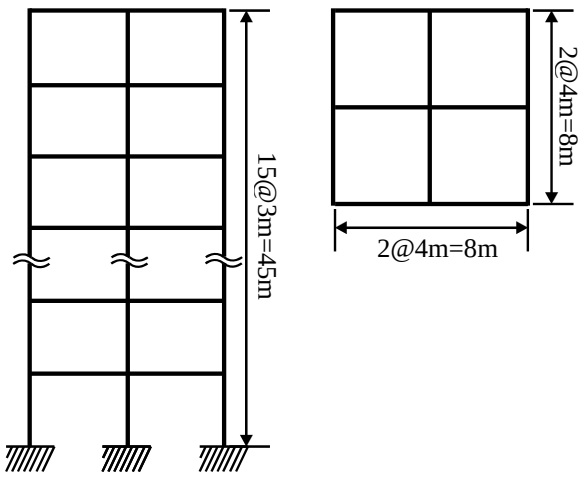


Figure 2.14 Error norm versus iteration number of the tenth eigenpair of the simply supported plate structure in the case of shift = λ_5

2.3.3 Three-Dimensional Framed Structure with Symmetric Cross-Section

The third example is a three-dimensional frame structure with symmetric cross-section. The geometric configuration and the material properties are shown in Figure 2.15. The structure is discretized by using 315 beam elements resulting in a system of dynamic equations with a total of 810 degrees of freedom. The consistent mass matrix is used for $\boldsymbol{M}$. The lowest ten eigenvalues of this structure are shown in Table 2.7. The eigenvalues of the model are distinct root or multiple ones.



(a) Elevation (b) Plan

$$A = 0.2787\ m^2,\quad I = 8.631\times 10^{-3}\ m^4$$
$$E = 2.068\times 10^{10}\ Pa,\quad \rho = 5.154\times 10^{2}\ kg/m^3$$

Figure 2.15 Three-dimensional frame structure with symmetric cross-section

Some results are shown in Table 2.8 and in Figures 2.16 to 2.21. The solution time for the two methods are summarized in Table 2.8. When a shift is on $1.01\lambda_4$, the subspace iteration method with shift and the proposed method obtain the required ten eigenpairs. However, when the shift is on $1.00001\lambda_4$ or on λ_4, the subspace iteration method with shift does not calculate the solutions while the proposed method finds the solutions. It shows that the iteration procedure for the proposed method can converge without any singularity even if the shift is exactly the same as the multiple eigenvalues, as analytically proved in Section 2.2.2.

For each solution method, the convergence of each eigenpair is depicted in Figures 2.16 to 2.21. Figures 2.16 and 2.17 show that when the shift is on $1.01\lambda_4$ the convergence of the proposed method is nearly equal to that of the subspace iteration method with shift. Figures 2.18 and 2.19 show that when the shift is on $1.00001\lambda_4$ the proposed method converges well without any singularity while the subspace iteration method with shift cannot converge due to the singularity. Figures 2.20 and 2.21 show that when the shift is exactly the same as the fourth eigenvalue the proposed method only converges well without any singularity. These results are the same as a shift is on $1.00001\lambda_4$. From the above results, it can be seen that the proposed method can choose a more exact shift than the subspace iteration method with shift, and thus the proposed method can be more computationally efficient.

Table 2.7 The lowest ten eigenvalues of the three-dimensional frame structure with symmetric cross-section

Mode number	Eigenvalues
1	0.1556E+03
2	0.1556E+03
3	0.3112E+03
4	0.1623E+04
5	0.1623E+04
6	0.2840E+04
7	0.5736E+04
8	0.5736E+04
9	0.8942E+04
10	0.1202E+05

Table 2.8 Solution time for the lowest ten eigenpairs of the three-dimensional frame structure with symmetric cross-section

Analysis methods	Shift = $1.01\lambda_4$	Shift = $1.00001\lambda_4$	Shift = λ_4
Subspace iteration method with shift	409.86 (1.00)	No solution	No solution
Proposed method	421.58 (1.03)	421.69	421.19

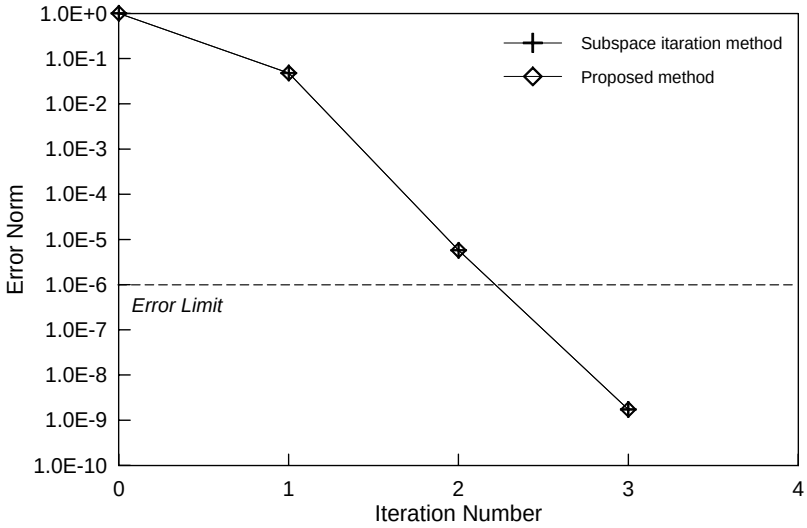


Figure 2.16 Error norm versus iteration number of the fourth eigenpair of the three-dimensional frame structure with symmetric cross-section in the case of shift = 1.01 λ_4

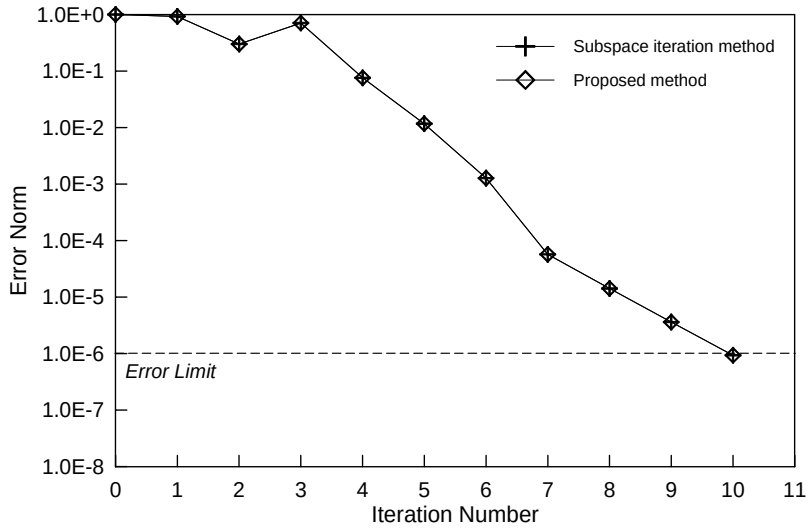


Figure 2.17 Error norm versus iteration number of the tenth eigenpair of the three-dimensional frame structure with symmetric cross-section in the case of shift = 1.01 λ_4

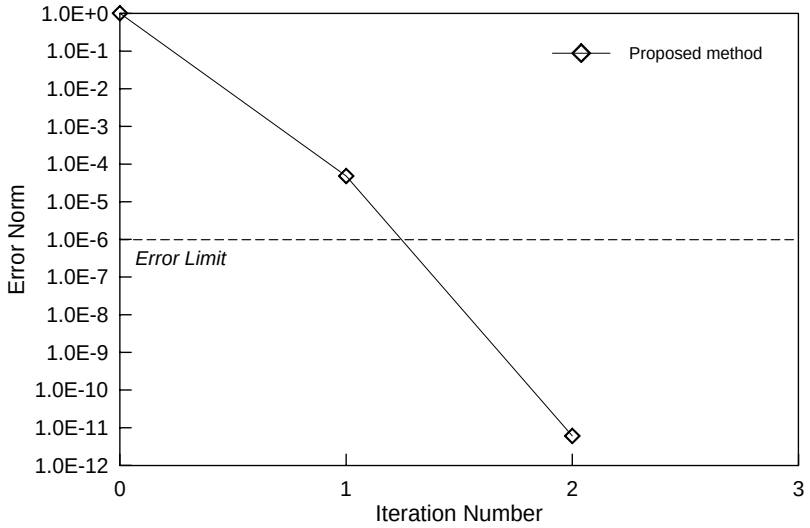


Figure 2.18 Error norm versus iteration number of the fourth eigenpair of the three-dimensional frame structure with symmetric cross-section in the case of shift = $1.00001\,\lambda_4$

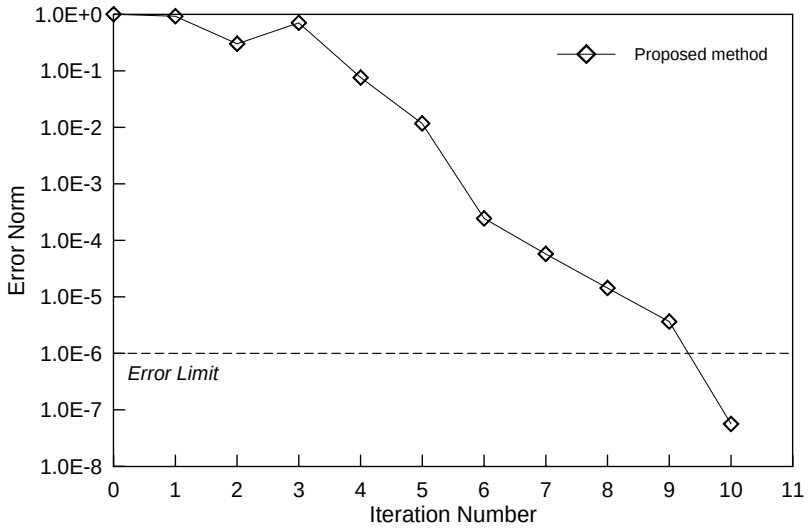


Figure 2.19 Error norm versus iteration number of the tenth eigenpair of the three-dimensional frame structure with symmetric cross-section in the case of shift = $1.00001\,\lambda_4$

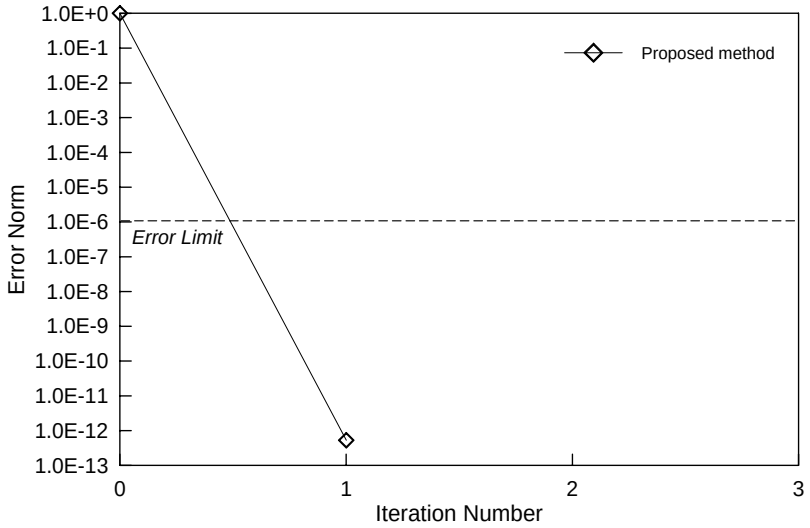


Figure 2.20 Error norm versus iteration number of the fourth eigenpair of the three-dimensional frame structure with symmetric cross-section in the case of shift = λ_4

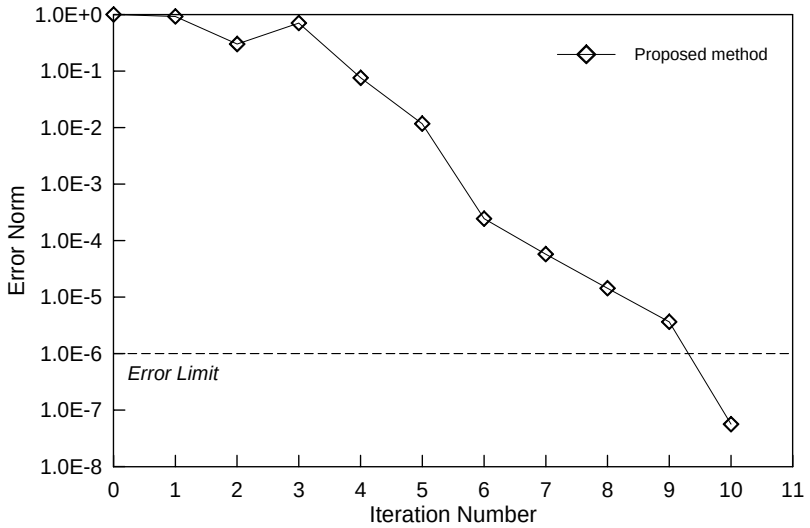
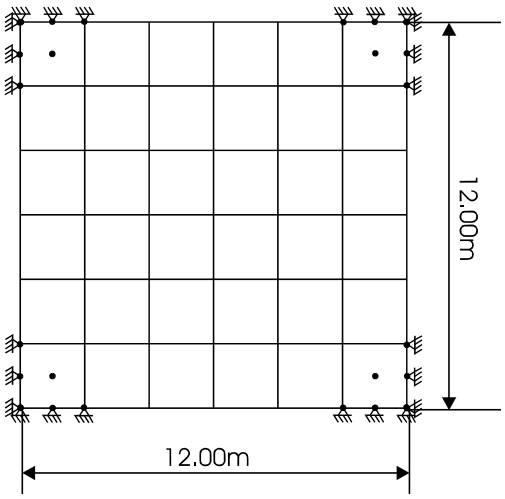


Figure 2.21 Error norm versus iteration number of the tenth eigenpair of the three-dimensional frame structure with symmetric cross-section in the case of shift = λ_4

2.3.4 Simply Supported Square Plate Structure

The fourth example is a simply supported square plate structure. Figure 2.22 shows the geometric configuration and material properties. The structure is discretized by using 36 shell elements (nine node per element) resulting in a system of dynamic equations with a total of 701 degrees of freedom. The consistent mass matrix is used for $\boldsymbol{M}$. The lowest ten eigenvalues of the simply supported square plate structure are shown in Table 2.9. The eigenvalues of the structure are distinct root or multiple ones.



$E=2.0\times10^{11}Pa$, $\rho=7.85\times10^{3}kg/m^{3}$

Poisson ratio $=0.3$, *Shell thickness* $=0.01m$

Figure 2.22 Simply supported square plate structure

Some results are shown in Table 2.10 and in Figures 2.23 to 2.28. The solution time for the two methods are summarized in Table 2.10. When a shift is on $1.01\lambda_2$, the subspace iteration method with shift and the proposed method obtain the required ten

eigenpairs. However, when the shift is on $1.0001\lambda_2$ or on λ_2, the subspace iteration method with shift does not calculate the solutions while the proposed method finds the solutions. It shows that the iteration procedure for the proposed method can converge without any singularity even if the shift is exactly the same as the multiple eigenvalues.

For each solution method, the convergence of each eigenpair is depicted in Figures 2.23 to 2.28. Figures 2.23 and 2.24 show that when the shift is on $1.01\lambda_2$ the convergence of the proposed method is nearly equal to that of the subspace iteration method with shift. Figures 2.25 and 2.26 show that when the shift is on $1.00001\lambda_2$ the proposed method converges well without any singularity while the subspace iteration method with shift can not converge due to the singularity. Figures 2.27 and 2.28 show that show that when the shift is exactly the same as the second eigenvalue, the proposed method only converges well without any singularity.

Table 2.9 The lowest ten eigenvalues of the simply supported square plate structure

Mode number	Eigenvalues
1	0.4435E+01
2	0.2914E+02
3	0.2914E+02
4	0.7367E+02
5	0.1305E+03
6	0.1305E+03
7	0.2087E+03
8	0.2087E+03
9	0.4010E+03
10	0.4418E+03

Table 2.10 Solution time for the lowest ten eigenpairs of the simply supported square plate structure

Analysis methods	Shift = $1.01\lambda_2$	Shift = $1.00001\lambda_2$	Shift = λ_2
Subspace iteration method with shift	720.95 (1.00)	No solution	No solution
Proposed method	751.05 (1.04)	750.64	751.25

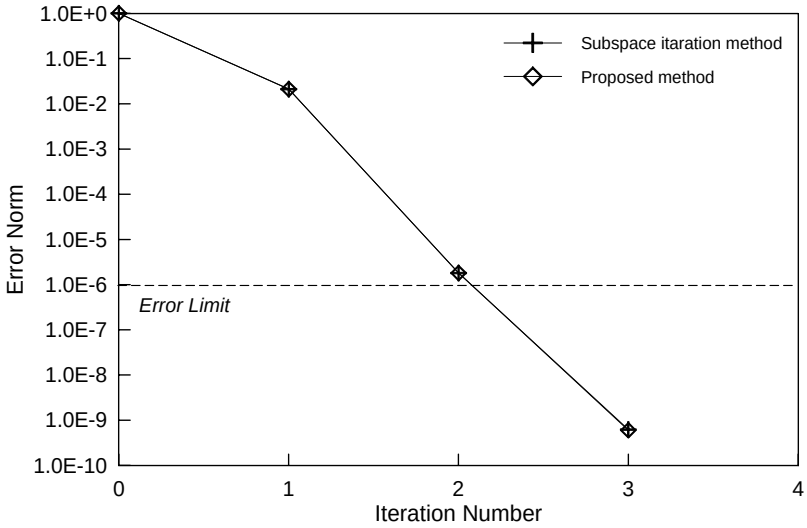


Figure 2.23 Error norm versus iteration number the second eigenpair of the simply supported square plate structure in the case of = 1.01 λ_2

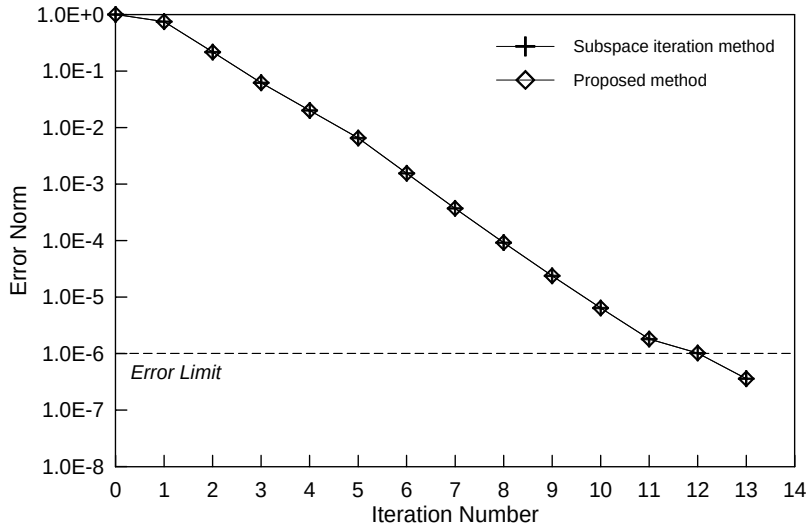


Figure 2.24 Error norm versus iteration number of the tenth eigenpair of the simply supported square plate structure in the case of shift = 1.01 λ_2

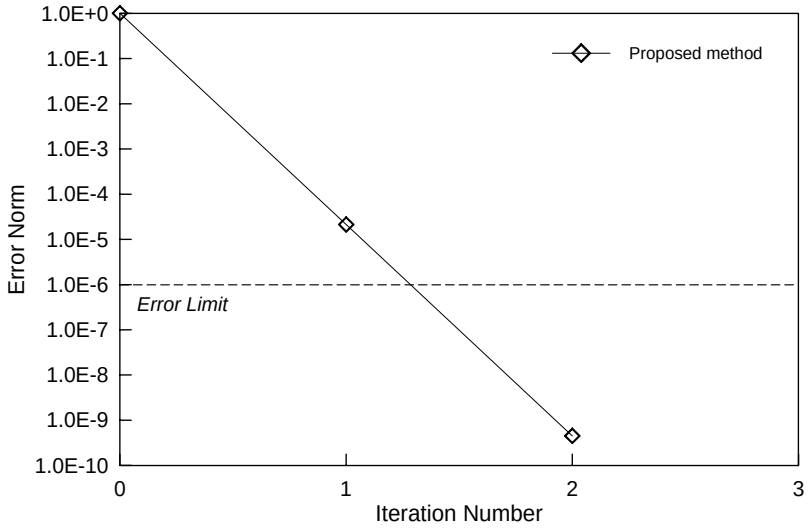


Figure 2.25 Error norm versus iteration number of the second eigenpair of the simply supported square plate structure in the case of shift = 1.00001 λ_2

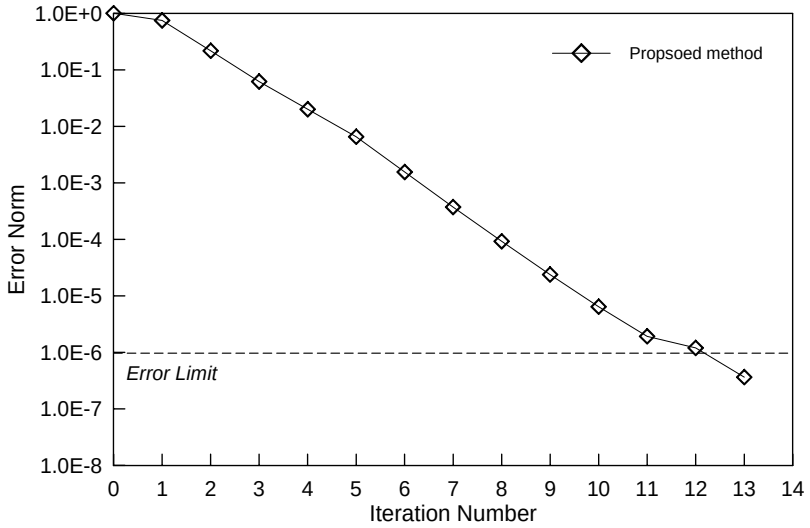


Figure 2.26 Error norm versus iteration number of the tenth eigenpair of the simply supported square plate structure in the case of shift = 1.00001 λ_2

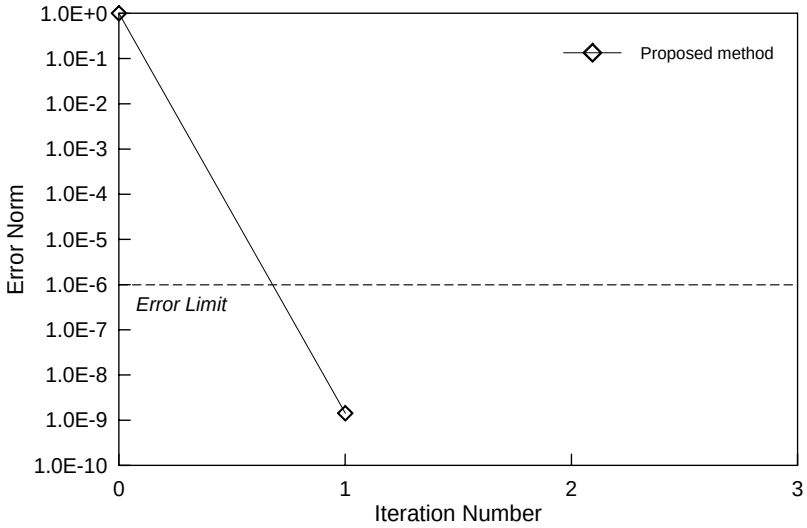


Figure 2.27 Error norm versus iteration number of the second eigenpair of the simply supported square plate structure in the case of shift = λ_2

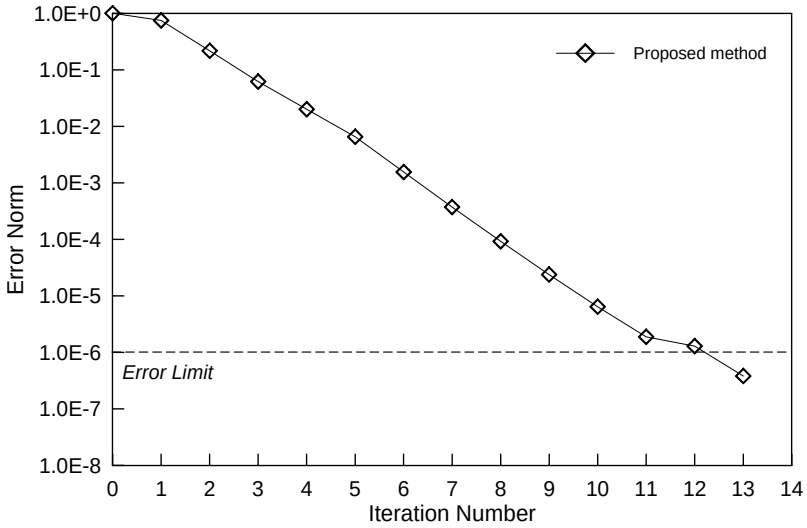


Figure 2.28 Error norm versus iteration number of the tenth eigenpair of the simply supported square plate structure in the case of shift = λ_2

2.4 Summary of Results

The proposed method is the improved the subspace iteration method with shift. By introducing the side conditions, a major difficulty of the conventional subspace iteration method with shift that is the singularity problem has been solved. As shown in numerical analysis, the characteristics of the proposed method identified as follows:

(1) The nonsingularity of the proposed method is always guaranteed, which is proved analytically; even if the shift is on or very close to a distinct eigenvalue or multiple ones, the proposed method can obtain the solutions without any singularity problem.

(2) The convergence rate of the proposed method is at least equal to that of the conventional subspace iteration method with shift, which is proved analytically, and the operation counts of the proposed method and the conventional subspace iteration method with shift are almost the same when the number of eigenpairs to be required is large.

To more effectively perform the eigenvalue analysis, the improved subspace iteration method with shift should be combined with the conventional subspace iteration method with shift. That is, in the case of a shift not close to an eigenvalue, the desired eigenpairs are obtained by the conventional method, and in the case of a shift close to an eigenvalue, the desired ones are calculated by the improved method. Then, the operation counts and solution time required in the eigenvalue analysis can be reduced. The above-mentioned idea is very simple, but the practical application is somewhat complicate. The research to integrate the above two methods is now in progress.

CHAPTER 3

TECHNIQUE FOR CHECKING MISSSED EIGENVALUES OF STRUCTURES WITH NONPROPORTIONAL DAMPING

3.1 Theoretical Background

3.1.1 Nonproportionally Damped System

The dynamic equation of motion of the damped system can be expressed as

$$\boldsymbol{M}\,\ddot{\boldsymbol{u}}(t)+\boldsymbol{C}\,\dot{\boldsymbol{u}}(t)+\boldsymbol{K}\,\boldsymbol{u}(t)=\boldsymbol{f}(t) \tag{3.1}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ the n by n mass, damping and stiffness matrices, respectively, $\boldsymbol{f}(t)$ the n by 1 external load vector, and $\ddot{\boldsymbol{u}}(t)$, $\dot{\boldsymbol{u}}(t)$ and $\boldsymbol{u}(t)$ the n by 1 acceleration, velocity and displacement vectors, respectively.

For symmetric and positive definite matrices, the eigenvalues and the corresponding eigenvectors of the system in equation (3.1) may be determined in a straightforward and efficient manner provided that the damping matrix is proportional or, more generally (Caughey and O'Kelly, 1965), provided that

$$\boldsymbol{C}\boldsymbol{M}^{-1}\boldsymbol{K}=\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{C}\,. \tag{3.2}$$

Systems for which equation (3.2) is not satisfied are called nonproportionally damped. The eigenproblem for nonproportionally damped system can be written as the following quadratic form:

$$\lambda_i^2\,\boldsymbol{M}\boldsymbol{\phi}_i+\lambda_i\,\boldsymbol{C}\boldsymbol{\phi}_i+\boldsymbol{K}\boldsymbol{\phi}_i=0 \quad (i=1,2,...,n) \tag{3.3}$$

where λ_i is the *i*th eigenvalue and $\boldsymbol{\phi}_i$ the corresponding eigenvector.

There are $2n$ eigenvalues for the nonproportionally damped system with n degrees of freedom and these occur either in real pairs or in complex conjugate pairs, depending upon whether they correspond to overdamped or underdamped modes. The eigenvalue analysis of this system becomes relatively expensive because complex arithmetic operations are required. Also, since complex symmetric matrices do not have many of the important properties of real symmetric matrices (Wilkinson, 1965), various properties that are used in the real arithmetic operations should be reviewed before these properties are applied to the nonproportionally damped system. The technique for checking the missed eigenvalues using the Sturm sequence property should be also reviewed whether it is still valid in the complex arithmetic operations or not. If the Sturm sequence property cannot be applied to the nonproportionally damped system, a new technique for checking the missed eigenvalues should be developed by selecting and modifying an appropriate mathematical property among various properties that are valid in the complex domain.

3.1.2 Review of Mathematical Properties

In this section, various mathematical properties have been reviewed thoroughly to select an appropriate property that can be applied to a technique for checking the missed eigenvalues in the eigenvalue analysis of structures with nonproportional damping (i.e., the eigenvalue analysis including damping).

First, the applicability of the Sturm sequence property to the eigenvalue analysis including damping has been investigated. Since the Sturm sequence property is used to determine the number of the real roots of a polynomial on the real axis (Henrici, 1974; Pearson, 1974; Jacobson, 1985), this property cannot be directly applied to the nonproportionally damped system having the complex eigenpairs. Tsai and Chen (1993)

proposed the extended Sturm sequence property that can determine the root distribution of a polynomial on some specified lines of the complex plane. However, this extended property cannot be applied to the nonproportionally damped system because it is very difficult to find the specified line of the complex plane in this case and the Sturm sequence cannot be formed by factorizing the considered matrix in the field of the complex arithmetic computation while the sequence can be formed easily by using the same procedure in the field of the real arithmetic computation. As seen from the above discussion, the Sturm sequence property cannot be extended to the nonproportionally damped system. Hence, among various mathematical properties that are valid in the complex plane, an appropriate property that can be applied to the eigenproblem including damping should be found.

The Gerschgorin's theorem (Meirovitch, 1980; Lancaster and Tismenetsky, 1985) gives the bounds for eigenvalues of a matrix. Since this theorem only requires the elements of the matrix, the operation procedure is very simple. However, by using the Gerschgorin's theorem the exact number of the eigenvalues that is inside an assigned domain cannot be counted, and so this theorem cannot be applied to the technique for checking the missed eigenvalues.

The Routh-Hurwitz criterion (Pearson, 1974; Lancaster and Tismenetsky, 1985) and the argument principle (Spiegel, 1964; Carrier et al., 1966; Korn and Korn, 1968; Henrici, 1974) are all the techniques that investigate the number of the complex zeros of a polynomial. Since the eigenvalues of the eigenproblem such as equation (3.3) are equal to the zeros of the characteristic polynomial of the eigenproblem as follows:

$$f(\lambda) = det(\lambda^2 \boldsymbol{M} + \lambda \boldsymbol{C} + \boldsymbol{K}) = a_0 + a_1\lambda + \cdots + a_{2n-1}\lambda^{2n-1} + a_{2n}\lambda^{2n} \qquad (3.4)$$

where $a_i\ (i = 0,1,...,2n)$ are the real coefficients, the missed eigenvalues of the eigenproblem may be checked by using the Routh-Hurwitz criterion and the argument principle.

However, the Routh-Hurwitz criterion has a significant disadvantage to be applied to a technique for counting the number of the eigenvalues in the eigenvalue analysis including damping. This criterion must know all the coefficients of the polynomial. It is impossible to find the coefficients of the characteristic polynomial of the large eigenproblem. This property cannot be, therefore, used to a technique for checking the missed eigenvalues. But on the other hand, the argument principle does not require the coefficients of the polynomial to check the number of zeros. It only uses the final value of the polynomial, and so this principle may be considered as the appropriate mathematical property that can be applied to the technique for checking the missed eigenvalues of structures with nonproportional damping. The argument principle is described in the next section in detail.

3.1.3 Argument Principle (Spiegel, 1964; Carrier et al., 1966; Korn and Korn, 1968; Henrici, 1974)

Let $f(z)$ be a polynomial of order n in z as follows:

$$f(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \tag{3.5}$$

where z is complex numbers and $a_i \ (i = 0,1,...,n)$ the real coefficients.

If $f(z)$ is analytic inside and on a simple closed contour S except for a finite number of poles inside the contour S (i.e., $f(z)$ is a meromorphic function), the following equation is introduced

$$\frac{1}{2\pi i} \oint_S \frac{f'(z)}{f(z)} dz = N - P \tag{3.6}$$

where N is the number of zeros and P the number of poles of $f(z)$ inside S,

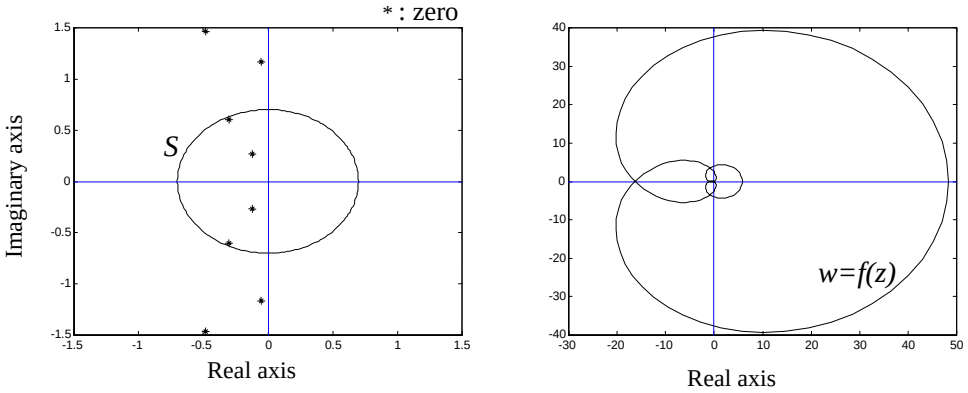
respectively, and $i = \sqrt{-1}$.

If a pole of $f(z)$ does not exist inside S, equation (3.6) reduces to the argument principle:

$$N = \frac{\Delta_S \theta}{2\pi} \tag{3.7}$$

where $\Delta_S \theta$ is the variation of the argument θ of $f(z)$ around the contour S, and a zero of multiplicity k is counted k times (Franklin et al. 1994).

Equation (3.7) means that $w = f(z)$ maps a moving point z describing the contour S into a moving point w which encircles the origin of the w plane N times if $f(z)$ has N zeros inside the contour S in the z plane. As seen from Figure 3.1, a moving point $w = f(z)$ encircles the origin of the w plane four times if $f(z)$ has four zeros inside the contour S in the z plane.



(a) z plane (b) w plane

Figure 3.1 Argument principle

3.2 Proposed Method

As seen from the previous section, a technique for checking the missed eigenvalues of structures with nonproportional damping can be developed by using the argument principle. Now, a technique for checking the missed eigenvalues that is applicable to the large eigenproblem including the damping matrix is proposed.

Let us consider the large structural system of order n as equation (3.1). By using the symbolic algebraic operations as in Section 3.1.3, it is impossible to find the characteristic polynomial $f(z)$ of the quadratic eigenproblem as equation (3.3):

$$f(\lambda) = det(\lambda^2 \boldsymbol{M} + \lambda \boldsymbol{C} + \boldsymbol{K}) = a_0 + a_1\lambda + \cdots + a_{2n-1}\lambda^{2n-1} + a_{2n}\lambda^{2n} \qquad (3.8)$$

where λ is complex numbers on the simple closed contour S. The inside of the contour S means the domain that is assigned to check the missed eigenvalues. Since the analytical solution in the field of the symbolic computation cannot be calculated, the numerical solution in the field of the complex arithmetic computation should be considered.

To develop a technique for checking the missed eigenvalues using the argument principle, the discretization of the simple closed contour S and the relationship between the characteristic polynomial and the factorized matrices by the $\boldsymbol{LDL^T}$ factorization should be considered. First, the simple closed contour S in Figure 3.1(a) is considered as the set of the checking points as described in Figure 3.2. And then, the $\boldsymbol{LDL^T}$ factorization process is performed at each checking point. Then, the argument θ_j at the checking point *j* can calculate as follows:

$$f(\lambda_j) = det(\lambda_j^2 \boldsymbol{M} + \lambda_j \boldsymbol{C} + \boldsymbol{K}) = det\, \boldsymbol{L}\, \boldsymbol{D}\, \boldsymbol{L}^T = \prod_{i=1}^{n} d_{ii} = r_j \angle \theta_j \qquad (3.9)$$

where d_{ii} is the diagonal elements of the diagonal matrix $\boldsymbol{D}$, and r_j and θ_j the magnitude and argument of the value $f(\lambda_j)$ in polar form, respectively.

The number of the eigenvalues inside the contour S is calculated by summing the variation of the argument of each checking point as follows:

$$N = \frac{\sum \Delta_S \theta_j}{2\pi} \tag{3.10}$$

where $\sum \Delta_S \theta_j$ is the total variation of the argument, that is, the sum of the variation of the argument at each checking point. As stated in Section 3.1.3, an eigenvalue of multiplicity k is counted k times (Franklin et al. 1994). Hence, the technique for checking the missed eigenvalues using the argument principle can be applied to an eigenvalue analysis of a structure with multiple eigenvalues.

The graphical representation of the proposed method is described in Figure 3.2. Since the eigenvalues are always complex conjugate pairs in the case of the eigenproblem of the underdamped system, the simple closed contour S is only considered in the second quarter-plane as in Figure 3.2. Hence, if the number of the calculated eigenvalues by performing an eigenvalue analysis is q, then the number of the eigenvalues considered for the checking process (p) is the half of the number of the above calculated eigenvalues (i.e., $p = q/2$). By extensive experience, we know that the appropriate number of the initial checking points should be at least $6p$ to check completely without any missed eigenpairs.

The process for checking the missed eigenpairs using the argument principle can be briefly described as follows. First, one checks from the origin to the maximum magnitude ($\rho = 1.001|\lambda_q|$) along the imaginary axis (① in Figure 3.2). At this time, one selects about $3p$ checking points, and then one performs the $\boldsymbol{LDL^T}$ factorizing process at each

checking point. And, one checks the second quarter-plane along the quadrant with radius ρ in counterclockwise (② in Figure 3.2). At this time, one selects about $3p$ checking points, and then one performs the $\boldsymbol{LDL^T}$ factorizing process at each checking point. The checking process in the real axis can skip because of no variation of the argument (③ in Figure 3.2).

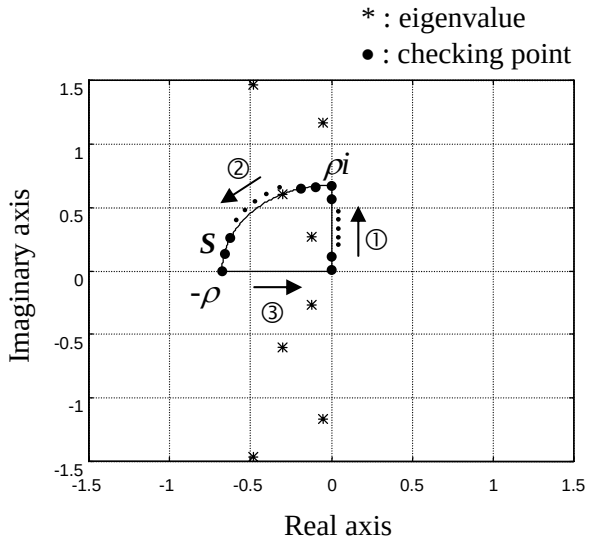


Figure 3.2 Process for checking the missed eigenvalues using the argument principle

If the decrease and the aggressive variation (about 200°) of the argument between two adjacent checking points may occur, the extra checking points between two adjacent checking points should be added. The total variation of the arguments is calculated by summing the argument of each checking point. Finally, it is checked the missed eigenvalues by comparing the total rotation number (N in equation (3.10)) with the number of the considered eigenvalues (p). The detailed algorithm of the proposed method is explained in Table 3.1.

Table 3.1 Algorithm of the proposed method

Step 1: Calculate the maximum magnitude, R.

- Select 1.001 times the absolute value of the *q*th eigenvalue ($\rho = 1.001|\lambda_q|$).

Step 2: Determine the number and the location of the initial checking points.

- *Number*: At least, select six times the number of the considered eigenvalues ($6p$).
- *Location*: Basically, divide into $6p$ equal parts.

Step 3: Perform the checking process.

1) Check the imaginary axis (the origin $\rightarrow \rho i$).

- Perform the $\boldsymbol{LDL^T}$ factorizing process at each checking point.
- Calculate the argument θ_j at each checking point.

2) Check the second quarter-plane ($\rho\angle 90° \rightarrow \rho\angle 180°$).

- Perform the $\boldsymbol{LDL^T}$ factorizing process at each checking point.
- Calculate the argument θ_j at each checking point.

Step 4: Analyze the variations of the arguments.

- If a decrease or aggressive variation of the argument occurs at a checking point, then go to Step 5 and if not, go to Step 6.

Step 5: Add the extra checking points.

- Go to Step 3.

Step 6: Check the missed eigenvalues.

- Calculate the total variation of the argument and the total rotation number.
- Compare the total rotation number (N in equation (3.10)) with the number of the considered eigenvalues (p).

3.3 Numerical Examples

To show the effectiveness of the proposed method, two numerical examples are analyzed. First, a simple spring-mass-damper system that has the correct analytical solutions is considered to verify that the proposed method can check exactly the missed eigenpairs for the eigenvalue analysis considering the damping matrix. And a three-dimensional frame structure is considered to verify that the proposed method can be applied to the large structural systems in practice.

3.3.1 Simple Spring-Mass-Damper System

The finite element discretization of the system results in a diagonal mass matrix, a tridiagonal damping and stiffness matrices of the following forms (Chen and Taylor, 1988)

$$\boldsymbol{M} = m\boldsymbol{I} \tag{3.11}$$

$$\boldsymbol{C} = \alpha \boldsymbol{M} + \beta \boldsymbol{K} \tag{3.12}$$

$$\boldsymbol{K} = k\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \tag{3.13}$$

where α and β are the damping coefficients of the Rayleigh damping. The analytical solutions can be resulted through following relationships

$$\lambda_{2i-1,2i} = -\xi_i \omega_i \pm j\omega_i \sqrt{1-\xi_i^2} \quad \text{for } i = 1, \ldots, n \tag{3.14}$$

$$\xi_i = \frac{1}{2}\left(\frac{\alpha}{\omega_i} + \beta\omega_i \right) \tag{3.15}$$

$$\omega_i = 2\sqrt{\frac{m}{k}} \sin \frac{2i-1}{2n+1} \frac{\pi}{2} \tag{3.16}$$

where ω_i and ξ_i are the undamped natural frequency and modal damping ratio, respectively.

A system with order 50 is used in analysis. k and m are 1, and the coefficients, α and β, of the Rayleigh damping are 0.05 and 0.5, respectively. The lowest six eigenvalues by analytical solutions are expressed as in Table 3.2. No missed eigenvalues exist in the lowest six eigenvalues because these are the analytical solutions.

Table 3.2 The lowest six eigenvalues by analytical solutions

Mode Number	Eigenvalues	
	Real	Imaginary
1	–0.02524	+0.01817
2	–0.02524	–0.01817
3	–0.02718	+0.08923
4	–0.02718	–0.08923
5	–0.03103	+0.15224
6	–0.03103	–0.15224

To verify that the proposed method can check exactly the missed eigenpairs for the eigenvalue analysis considering the damping matrix, the checking process is performed. First, since the second quarter-plane in the complex plane is only considered, the number of the considered eigenpairs is three ($p = 3$). And, the maximum magnitude is calculated by 1.001 times the absolute value of the sixth eigenvalue ($\rho = 1.001|\lambda_6|$ =0.1555). To select the initial checking points, the line from the origin to ρi on the imaginary axis and the quadrant with radius ρ in the second quarter-plane are divided into nine equal parts, respectively. The total number of the initial checking points is, therefore, 18 ($6p = 6 \times 3 = 18$).

Then, the factorizing process at each checking point is performed, and the variation of the argument at each one is reviewed. After the additional checking points are selected through the review on the results of the first checking process, the second and final checking process is performed. The results of each factorizing process are described in Table 3.3 and the change of the total variation of the argument is described in Figure 3.3.

Since the total variation of the argument is 1,080° as seen from Table 3.3, the total number of rotations is as follows:

$$N = \frac{\sum \Delta_S \theta_j}{2\pi} = \frac{1{,}080^\circ}{360^\circ} = 3.$$

Finally, the missed eigenpairs is checked by comparing the number of rotations with the number of the calculated eigenvalues. Since the number of rotations and the number of the considered eigenvalues are all the same (i.e., $N = p = 3$), the missed eigenvalues do not exist in the simple closed contour S.

Table 3.3 The arguments and the variations of the arguments

Initial checking process				Second checking process				Final checking process				$\sum \Delta_S \theta_j$
λ_j	θ_j	$\Delta_S \theta_j$	*Y/N*	λ_j	θ_j	$\Delta_S \theta_j$	*Y/N*	λ_j	θ_j	$\Delta_S \theta_j$	*Y/N*	
origin	0.00	-	-									0.00
ρ/9	89.1	89.1	*N*									89.1
2ρ/9	172.1	83.0	*N*									172.1
3ρ/9	239.9	67.8	*N*									239.9
4ρ/9	305.1	65.2	*N*									305.1
5ρ/9	19.7	74.6	*N*									379.7
6ρ/9	98.1	78.4	*N*									458.1
7ρ/9	168.0	69.9	*N*									528.0
8ρ/9	237.9	69.9	*N*									597.9
ρ	314.6	76.7	*N*									674.6
ρ∠100°	319.6	5.0	*N*									679.6
								ρ∠100.5°	321.0	1.4	*N*	681.0
								ρ∠101.0°	322.2	1.2	*N*	682.2
								ρ∠101.5°	24.9	62.7	*N*	744.9
				ρ∠102°	**132.8**	**173.2**	***Y***	*ρ∠102°*	132.8	107.9	*N*	852.8
				ρ∠104°	137.3	4.5	*N*					857.3
				ρ∠106°	138.1	0.8	*N*					858.1
				ρ∠108°	139.1	1.0	*N*					859.1
ρ∠110°	**140.5**	**180.9**	***Y***	*ρ∠110°*	140.5	1.4	*N*					860.5
ρ∠120°	152.6	12.1	*N*									872.6
ρ∠130°	174.0	21.4	*N*									894.0
ρ∠140°	202.1	28.1	*N*									922.1
ρ∠150°	236.1	34.0	*N*									956.1
ρ∠160°	274.7	38.6	*N*									994.7
ρ∠170°	316.5	41.8	*N*									1,016.5
ρ∠180°	0.0	43.5	*N*									1,080.0

where $0° \le \theta_j < 360°$, $\Delta_S \theta_j = \theta_j - \theta_{j-1}$, and '***Y***' means that the additional checking points are required and '*N*' means that the additional ones are not required.

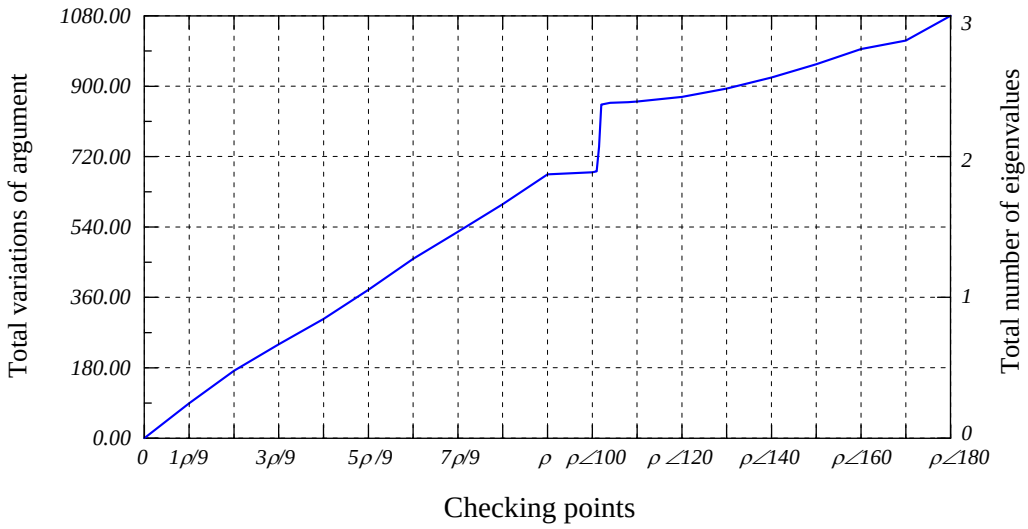
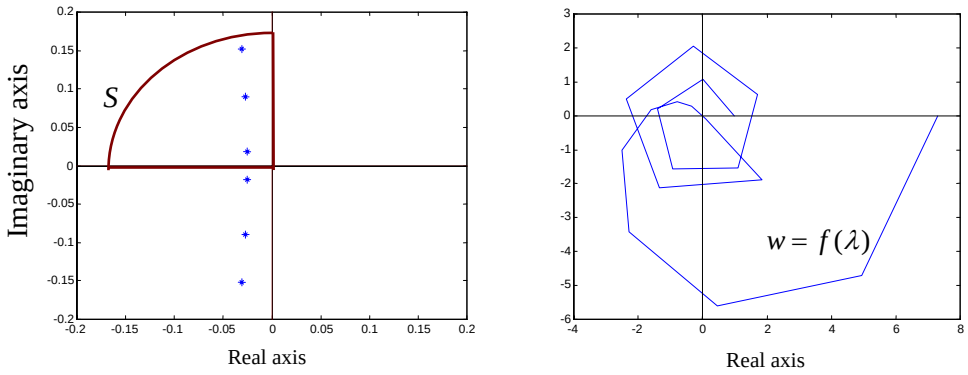


Figure 3.3 Change of total variation of argument and total number of eigenvalues with checking points

To describe the above results in the graphical representation such as Figure 3.1, Figure 3.4 is introduced. Since three eigenvalues exist inside the simple closed contour S in the second quarter-plane of the λ plane (Figure 3.4(a)), the function $f(\lambda)$ encircles the origin of the w plane three times in counterclockwise (Figure 3.4(b)). That is, the proposed method is a technique that can check the missed eigenpairs in the eigenvalue analysis including the damping matrix, because the technique can exactly find the number of the eigenvalues inside the simple closed contour.

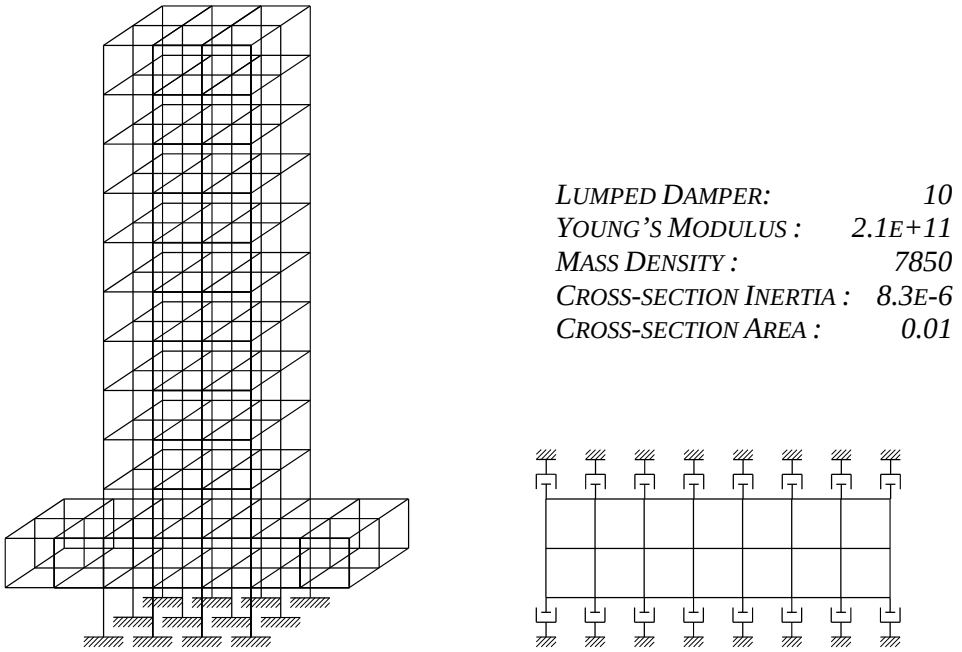


(a) λ plane (b) w plane

Figure 3.4 Graphical representation of the proposed method

3.3.2 Three-Dimensional Frame Structure with Concentrated Dampers

In this example, a large three-dimensional frame structure with concentrated dampers is presented. Two layers of the foundation are damped as shown in Figure 3.5. The system could be considered as a representative of a control system or a passively damped space structure. The model has 1,008 degrees of freedom. The material and cross-sectional properties are $E = 2.1\times10^{11}$, $I = 8.3\times10^{-6}$ and $\rho = 7{,}850$. The consistent mass matrix is used to define $\boldsymbol{M}$. The damping matrix $\boldsymbol{C}$ consists of the Rayleigh damping and concentrated dampers. The Rayleigh coefficient α is 0.0306 and β is 0.1016. The damping for each damper is 10.



(a) Three-dimensional frame structure. (b) Damping from two-layer foundation

Figure 3.5 Three-dimensional frame structure with concentrated dampers

The lowest ten eigenvalues are calculated by the Lanczos method developed by Kim and Lee (1999) as in Table 3.4. If the lowest p eigenvalues are desired, the eigenproblem reduced by $2p$ Lanczos vectors is generally solved. To obtain the lowest ten eigenvalues, thus, the Lanczos method was performed using twenty Lanczos vectors. Since the missed eigenpairs may exist in the required ones, it is necessary to check the missed ones.

Table 3.4 The lowest ten eigenvalues by the Lanczos method using twenty Lanczos vectors

Mode Number	Eigenvalues	
	Real	Imaginary
1	–0.0304	+3.0300
2	–0.0304	–3.0300
3	–0.0309	+3.0900
4	–0.0309	–3.0900
5	–0.0374	+3.6581
6	–0.0374	–3.6581
7	–0.1420	+8.7269
8	–0.1420	–8.7269
9	–0.1445	+8.8222
10	–0.1445	–8.8222

To verify that the proposed method can be applied to the large structural systems in practice, the checking process is performed. First, since the second quarter-plane in the complex plane is only considered, the number of the considered eigenvalues is five ($p = 5$). And, the maximum magnitude is calculated by 1.001 times the absolute value of

the tenth eigenvalue ($\rho = 1.001|\lambda_{10}|$ =8.8323). To select the initial checking points, the line from the origin to ρi on the imaginary axis and the quadrant with radius ρ in the second quarter-plane are divided into nine equal parts, respectively. Also, since all the eigenvalues are close to the imaginary axis, the severe variation of the argument is expected when the checking process is performed at the checking point close to each eigenvalue. To avoid the above aggressive variation of the argument, the intervals as $5\rho/15 \sim 6\rho/15$, $6\rho/15 \sim 7\rho/15$, $14\rho/15 \sim \rho$ and $\rho\angle 90 \sim \rho\angle 93$ that are close to each eigenvalue are subdivided, respectively.

Then, the factorizing process at each checking point is performed, and the variation of the argument at each one is reviewed. After the additional checking points are selected through the review on the results of the first checking process, the final checking process is performed. The results of each factorizing process are described in Table 3.5 and the change of the total variation of the argument is described in Figure 3.6.

Since the total variation of the argument is 2,160° as seen from Table 3.5, we conclude that the total number of rotations is

$$N = \frac{\sum \Delta_S \theta_j}{2\pi} = \frac{2{,}160^\circ}{360^\circ} = 6.$$

Finally, we check the missed eigenvalues by comparing the number of rotations with the number of the calculated eigenvalues. Since the number of rotations is six ($N = 6$) and the number of the considered eigenvalues five ($p = 5$), the number of the missed eigenvalues in the simple closed contour S is one ($N - p = 1$).

To verify this result, the eigenvalue analysis is performed again. By the Lanczos method using one hundred Lanczos vectors, the more exact eigenvalues are obtained. As seen from Table 3.6, the six eigenvalues exist inside the simple closed contour S with radius ρ. One eigenvalue pair (i.e., the ninth and tenth eigenvalues in Table 3.6) is missed in the previous eigenvalue analysis (i.e., the Lanczos method using twenty Lanczos

vectors).

In view of the results so far achieved, it is verified that the proposed method is a technique for exactly checking the missed eigenvalues of large-scale structures with nonproportional damping.

Table 3.5 The arguments and the variations of the arguments

Initial checking process				Final checking process				$\sum \Delta_S \theta_j$
λ_j	θ_j	$\Delta_S \theta_j$	*Y/N*	λ_j	θ_j	$\Delta_S \theta_j$	*Y/N*	
origin	0.00	-	-					0.00
ρ/15	111.6	110.6	*N*					111.6
2ρ/15	223.5	111.9	*N*					223.5
3ρ/15	335.8	112.3	*N*					335.8
4ρ /15	89.9	114.1	*N*					449.9
5ρ/15	229.1	139.2	*N*					589.1
5.2ρ/15	44.5	175.4	*N*					764.5
5.4ρ/15	213.1	168.6	*N*					933.1
5.6ρ/15	252.5	39.4	*N*					972.5
5.8ρ/15	282.6	30.1	*N*					1,002.6
6ρ/15	315.3	32.7	*N*					1,035.3
6.2ρ/15	41.3	86.0	*N*					1,121.3
6.4ρ/15	147.1	105.8	*N*					1,227.1
6.6ρ/15	179.6	32.5	*N*					1,259.6
6.8ρ/15	205.6	26.0	*N*					1,285.6
7ρ/15	229.9	24.3	*N*					1,309.9
8ρ/15	345.7	115.8	*N*					1,425.7
9ρ/15	99.4	113.7	*N*					1,539.4
10ρ/15	213.0	113.6	*N*					1,653.0
11ρ/15	327.4	114.4	*N*					1,767.4
12ρ/15	83.4	116.0	*N*					1,883.4
13ρ/15	203.8	120.4	*N*					2,003.8
14ρ/15	342.6	138.8	*N*					2,142.6
14.2ρ/15	19.6	37.0	*N*					2,179.6
14.4ρ/15	68.0	48.4	*N*					2,228.0
14.6ρ/15	143.7	75.7	*N*					2,303.7
14.8ρ/15	264.9	121.2	*N*					2,424.9
ρ	40.9	136.0	*N*					2,560.9
				ρ∠90.5°	80.7	39.8	*N*	2,600.7
ρ∠91°	**314.5**	**273.6**	***Y***	*ρ∠91°*	314.5	233.8	*N*	2,834.5
ρ ∠92°	57.4	102.9	*N*					2,937.4
ρ∠93°	84.0	26.6	*N*					2,964.0

where $0° \le \theta_j < 360°$, $\Delta_S \theta_j = \theta_j - \theta_{j-1}$, and '***Y***' means that the additional checking points are required and '*N*' means that the additional ones are not required.

Table 3.5 The arguments and the variations of the arguments (*continued*)

Initial checking process				Final checking process				$\sum \Delta_S \theta_j$
λ_j	θ_j	$\Delta_S \theta_j$	*Y/N*	λ_j	θ_j	$\Delta_S \theta_j$	*Y/N*	
$\rho\angle 96°$	94.2	10.2	*N*					2,974.2
				$\rho\angle 99°$	83.2	-11.0	*N*	2,963.2
$\rho\angle 102°$	**66.4**	**332.2 *or* -27.8**	***Y***	$\rho\angle 102°$	66.4	-16.8	*N*	2,946.4
				$\rho\angle 105°$	46.6	-19.8	*N*	2,926.6
$\rho\angle 108°$	**25.0**	**318.6 *or* -41.4**	***Y***	$\rho\angle 108°$	25.0	-21.6	*N*	2,905.0
				$\rho\angle 111°$	2.1	-22.9	*N*	2,882.1
$\rho\angle 114°$	**338.1**	**313.1 *or* -46.9**	***Y***	$\rho\angle 114°$	338.1	-24.0	*N*	2,858.1
				$\rho\angle 117°$	313.3	-24.8	*N*	2,833.3
$\rho\angle 120°$	**287.6**	**309.5 *or* -50.5**	***Y***	$\rho\angle 120°$	287.6	-25.7	*N*	2,807.6
				$\rho\angle 123°$	261.1	-26.5	*N*	2,781.1
$\rho\angle 126°$	**233.8**	**306.2 *or* -53.8**	***Y***	$\rho\angle 126°$	233.8	-27.3	*N*	2,753.8
				$\rho\angle 129°$	205.7	-28.1	*N*	2,725.7
$\rho\angle 132°$	**176.8**	**303.0 *or* -57.0**	***Y***	$\rho\angle 132°$	176.8	-28.9	*N*	2,696.8
				$\rho\angle 135°$	147.3	-29.5	*N*	2,667.3
$\rho\angle 138°$	**117.0**	**300.2 *or* -59.8**	***Y***	$\rho\angle 138°$	117.0	-30.3	*N*	2,637.0
				$\rho\angle 141°$	86.0	-31.0	*N*	2,606.0
$\rho\angle 144°$	**54.4**	**297.4 *or* -62.6**	***Y***	$\rho\angle 144°$	54.4	-31.6	*N*	2,574.4
				$\rho\angle 147°$	22.1	-32.3	*N*	2,542.1
$\rho\angle 150°$	**349.3**	**294.9 *or* -65.1**	***Y***	$\rho\angle 150°$	349.3	-32.8	*N*	2,509.3
				$\rho\angle 153°$	316.0	-33.3	*N*	2,476.0
$\rho\angle 156°$	**282.1**	**292.8 *or* -67.2**	***Y***	$\rho\angle 156°$	282.1	-33.9	*N*	2,442.1
				$\rho\angle 159°$	247.8	-34.3	*N*	2,407.8
$\rho\angle 162°$	**213.2**	**291.1 *or* -68.9**	***Y***	$\rho\angle 162°$	213.2	-34.6	*N*	2,373.2
				$\rho\angle 165°$	178.2	-35.0	*N*	2,338.2
$\rho\angle 168°$	**142.9**	**289.7 *or* -70.3**	***Y***	$\rho\angle 168°$	142.9	-35.3	*N*	2,302.9
				$\rho\angle 171°$	107.3	-35.6	*N*	2,267.4
$\rho\angle 174°$	**71.7**	**288.8 *or* -71.2**	***Y***	$\rho\angle 174°$	71.7	-35.6	*N*	2,231.7
				$\rho\angle 177°$	35.9	-35.8	*N*	2,195.9
$\rho\angle 180°$	**0.00**	**288.3 *or* -71.7**	***Y***	$\rho\angle 180°$	0.00	-35.9	*N*	2,160.0

where $0° \le \theta_j < 360°$, $\Delta_S \theta_j = \theta_j - \theta_{j-1}$, and '***Y***' means that the additional checking points are required and '*N*' means that the additional ones are not required.

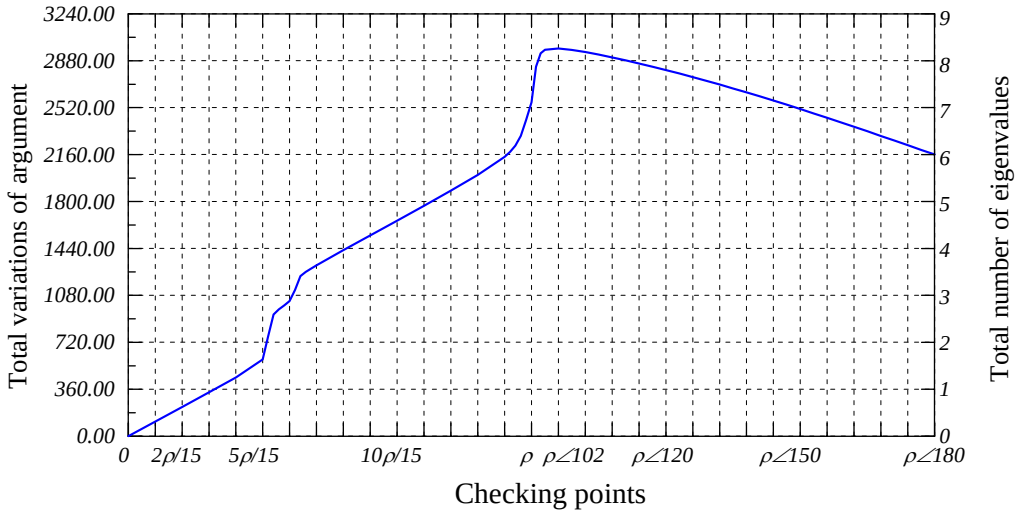


Figure 3.6 Change of total variation of argument and total number of eigenvalues with checking points

Table 3.6 The lowest twelve eigenvalues by the Lanczos method using one hundred Lanczos vectors

Mode Number	Eigenvalues	
	Real	Imaginary
1	–0.0304	+3.0301
2	–0.0304	–3.0301
3	–0.0309	+3.0901
4	–0.0309	–3.0901
5	–0.0374	+3.6581
6	–0.0374	–3.6581
7	–0.1427	+8.6586
8	–0.1427	–8.6586
9	–0.1404	+8.7465
10	–0.1404	–8.7465
11	–0.1441	+8.8290
12	–0.1441	–8.8290

3.4 Summary of Results

A technique for checking the missed eigenvalues of structures with non-proportional damping is obtained by using the argument principle. From the results of the numerical examples, the concluding remarks are summarized as follows.

(1) After the comprehensive review on various mathematical properties, it has been concluded that the argument principle is only applicable to a technique for checking the missed eigenvalues for the nonproportionally damped system.

(2) A checking technique that can apply to the nonproportionally damped system by using the argument principle is developed, and the effectiveness of the proposed method is verified by analyzing the numerical examples.

The technique for checking the missed eigenvalues using the Sturm sequence property only requires one factorization process at one checking point. On the other hand, the proposed method requires many factorization processes at many checking points. This cannot be inevitable, because the proposed method is executed in the complex plane whereas the checking technique using the Sturm sequence property is performed on the real axis. The proposed method, therefore, has a shortcoming that needs a large number of operation counts. To more effectively apply the proposed method to the practical problem, the research to reduce the operation counts of the proposed method should be performed.

CHAPTER 4

CONCLUSIONS

4.1 Summary and Conclusions

This study proposes a numerically stable eigenproblem solution method by improving the well-known subspace iteration method with shift and a technique for checking the missed eigenvalues of structures with nonproportional damping by using the argument principle.

First, the characteristics of the improved subspace iteration method with shift identified by the analytical and the numerical results from examples are summarized as follows:

(1) The nonsingularity of the proposed method is always guaranteed, which is proved analytically; even if the shift is on a distinct eigenvalue or multiple ones, the proposed method can obtain the desired eigenvalues and the corresponding eigenvectors without any singularity problem.

(2) The convergence rate of the proposed method is at least equal to that of the subspace iteration method with shift, which is proved analytically, and the operation counts of the proposed method and the subspace iteration method with shift are almost the same when a large number of eigenpairs are required.

And, the characteristics of the technique for checking the missed eigenvalues of structures with nonproportional damping identified by analyzing the numerical examples are summarized as follows:

(1) Through the comprehensive review on various mathematical properties, it has been verified that the argument principle is only applicable to a technique for checking the missed eigenvalues of structures with nonproportional damping.

(2) By using the argument principle, a technique for checking the missed eigenvalues that can be applied to the nonproportionally damped system is developed, and by analyzing the numerical examples, it is verified that the proposed method can check exactly the missed eigenvalues and can be applicable to the large eigenproblem.

4.2 Recommendations for Further Study

The improved subspace iteration method with shift presented in this dissertation is not only numerically stable but also efficient in the case of a shift close to an eigenvalue. However, in the case of a shift not close to an eigenvalue, the improved method needs more operations than the conventional subspace iteration method with shift. To more effectively perform the eigenvalue analysis, the improved subspace iteration method with shift should be combined with the conventional subspace iteration method with shift. That is, in the case of a shift not close to an eigenvalue, the desired eigenpairs are obtained by the conventional method, and in the case of a shift close to an eigenvalue, the desired ones are calculated by the improved method. Then, the operation counts and solution time required in the eigenvalue analysis can be reduced. The above-mentioned idea is very simple, but the practical application is somewhat complicate. The research to integrate the above two methods is now in progress.

The technique for checking the missed eigenvalues of structures with non-proportional damping that has first developed in this dissertation shall be studied further for practical applications. The technique for checking the missed eigenvalues using the Sturm sequence property, which is applied to the eigenvalues for the undamped or proportionally damped system, only requires one factorization process at one checking point. On the other hand, the technique for checking the missed eigenvalues using the argument principle requires many factorization processes at many checking points. This cannot be inevitable, because the proposed method is executed in the complex plane whereas the technique using the Sturm sequence property is performed on the real axis. The proposed method, therefore, has a shortcoming that needs relatively a large number of operation counts. To more efficiently apply the proposed method to the practical problem, the research to reduce the operation counts of the proposed method should be performed.

요 약 문

부분공간 반복법 개선 및 비비례감쇠 구조물의 누락된 고유치 검사 기법 개발

구조물의 정확한 동적응답을 구하기 위해서나 공진 현상을 피하기 위해서는, 수치적으로 안정하고 효율적인 고유치 해법 및 고유치 해석을 통해 구해진 고유짝에 누락된 고유짝이 존재하는지 여부를 정확하게 검사할 수 있는 기법이 필요하다. 본 논문에서는 부분공간 반복법의 개선을 통해 개발된 수치적으로 안정한 고유치 해법과 비비례감쇠를 갖는 구조물의 누락된 고유치 검사 기법을 제안하였다.

부분공간 반복법은 현재까지 대형 고유치문제를 계산하는데 가장 효율적이라고 알려진 고유치해법이다. 쉬프팅 기법을 사용한 부분공간 반복법의 가장 큰 문제점은 특이성 문제 때문에 어떤 고유치에 매우 근접한 쉬프트를 사용할 수 없다는 것으로, 이에 따라 수렴성이 낮아지는 결과를 준다. 본 연구에서는, 이상과 같은 특이성 문제를 부가조건식을 도입하여 수렴성 저하 없이 해결하였다. 제안방법은 쉬프트가 어떤 단일근 또는 중복근 상에 존재하는 경

우일 지라도 항상 비특이성을 만족한다. 제안 방법의 비특이성은 해석적으로 증명하였다. 제안방법의 수렴성은 최소한 기존의 쉬프트를 갖는 부분공간 반복법의 수렴성과 같고, 이것도 해석적으로 증명하였다. 제안방법과 기존 부분공간 반복법의 연산회수는 많은 고유짝을 구하는 경우에 거의 비슷하다. 제안방법의 효용성을 증명하기 위해서 네가지 예제에 대한 수치해석을 수행하였다. 먼저 3 차원 뼈대 구조물 및 단순지지된 사각 평판 구조물과 같이 단일근만을 갖는 구조물에 대해서 해석을 수행하였다. 그리고나서, 대칭단면을 갖는 3 차원 뼈대 구조물과 단순지지된 정사각 평판 구조물과 같이 중복근을 갖는 구조물에 대해 해석을 수행하였다.

부분공간 반복법, Lanczos 방법과 같은 대부분의 고유치해석 방법들은 구하고자 하는 고유짝 중의 일부를 누락시킬 수 있다. 왜냐하면 이러한 고유치해법들은 어떤 구조물의 전체 고유벡터 집합을 구하는 것이 아니라 이 집합에서 저차의 일부분만을 계산하기 때문이다. 따라서, 모드중첩법을 이용하여 정확한 동적응답을 얻기 위해서는 고유치해석 수행 후에 누락된 고유치를 검사하는 과정이 반드시 수행되어야만 한다. 비감쇠 또는 비례 감쇠 시스템의 경우, 즉 감쇠를 고려하지 않은 고유치해석의 경우에는 Sturm 수열 성질을 이용하여 누락된 고유치를 검사할 수 있다. 대부분의 고유치해법들이 이 기법을 채택하고 있다. 그러나, 지반-구조물 상호작용 문제, 구조물의 진동제어 문제,

복합재료 구조물과 같은 비비례 감쇠 시스템의 경우, 즉 감쇠를 고려한 고유치해석의 경우에는 누락된 고유치를 검사하는 기법이 아직까지 개발되어 있지 않다. 본 연구에서는, Sturm 수열 성질, Gerschgorin 정리, Routh-Hurwitz 기준 및 편각의 원리와 같은 다양한 수학 성질들을 자세히 검토한 후, 감쇠를 고려한 고유치 해석에서 누락된 고유치를 검사할 수 있는 기법을 제안하였다. 이 기법은 편각의 원리를 이용하였다. 제안방법의 효용성을 검증하기 위하여, 단순 스프링-질량-감쇠기 시스템과 집중 감쇠기가 부착된 3 차원 뼈대 구조물에 대해서 수치해석을 수행하였다.

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감사의 글

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CURRICULUM VITAE

Name	:	**Hyung-Jo Jung**
Date of Birth	:	September 11, 1971
Place of Birth	:	Chonbuk, Korea

EDUCATION

Ph.D.	Civil Engineering Korea Advanced Institute of Science and Technology, Korea March 1995 – Present
M.S.	Civil Engineering Korea Advanced Institute of Science and Technology, Korea March 1993 – February 1995
B.S.	Mechanical Engineering Korea Advanced Institute of Science and Technology, Korea March 1989 – February 1993

EXPERIENCE

March 1995 to Present	Graduate Research Assistant, Department of Civil Engineering, KAIST, Korea

LIST OF PUBLICATIONS

Dissertations:

1. "An Efficient Free Vibration Analysis of Structures with Multiple or Close Natural Frequencies," *M.S. Dissertation, Department of Civil Engineering,* Korea Advanced Institute of Science and Technology, Taejon, Korea, February 1995.

2. "Improvement of Subspace Iteration Method and Development of Technique for Checking Missed Eigenvalues of Structures with Nonproportional Damping," *Ph.D. Dissertation, Department of Civil Engineering,* Korea Advanced Institute of Science and Technology, Taejon, Korea, August 1999.

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