Modified Sturm Sequence Property for Damped Systems

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ABSTRACT

Most of the eigenvalue analysis methods for the undamped or proportionally damped systems use the well-known Sturm sequence property to check the missed eigenvalues when only a set of the lowest modes is to be used for large structures.

However, in the case of the non-proportionally damped systems such as the soil-structure interaction system, the structural control system and the composite structures, no counterpart of the Sturm sequence property for undamped systems has been developed yet. Hence, when some important modes are missed for those systems, it may lead to poor results in dynamic analysis.

In this study, a numerical method for calculating the number of eigenvalues in an open disk of arbitrary radius for the eigenproblem with a damping matrix is proposed by applying Gleyse’s theorem. To verify the applicability of the proposed method, two numerical examples are considered.
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CHAPTER 1

INTRODUCTION

1.1 Background

To obtain the dynamic response of a large civil structure, it is economic and efficient to superpose the results of a few lowest modes. Therefore, there have been proposed many eigensolution techniques which can find only a set of the lowest modes. The Lanczos and subspace methods are belong to this type of technique. In these techniques, however, some important modes can be missed in the calculation process, which may lead to poor results in dynamic analysis. Hence, a checking technique for missed eigenvalues is required in finding the missed ones. For the case of the undamped or proportionally damped systems, they can be easily found by using the well-known Sturm sequence property (Meirovitch 1980; Hughes 1987; Petyt 1990; Bathe 1996).

However, in the case of the non-proportionally damped systems such as the soil-structure interaction system, the structural control system and composite structures and so on, no counterpart of the Sturm sequence property for undamped systems has been developed yet (Newland 1989). Hence, when some important modes are missed for those systems, it may lead to poor results in dynamic analysis. A number of researchers (Rajakumar 1993; Kim and Lee 1999) has been performed to solve the eigenproblem with
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the damping matrix, whereas there have been few studies on a technique to calculate the number of eigenvalues in this case in the open literature.

Tsai and Chen (1993) proposed the extended Sturm sequence property that can determine the root distribution of a polynomial on some specified lines of the complex plane. However, this extended property can not be applied to the non-proportionally damped system because the calculated lowest eigenvalues are ordered according to their distances form the origin in the complex plane and also because the extended Sturm sequence property needs a region surrounded by lines in the complex plane. So to apply this property, we must divide the boundary of the region into many lines to reshape the boundary to a circle. And also, the extended Sturm sequence property needs some symbolic operations, which are very difficult to realize numerically. Jung et al. (2001) proposed a technique of checking missed eigenvalues for eigenproblem with damping matrix using argument principle. This method requires the selection of checking points and the $LDL^T$ factorization of the characteristic polynomial at those points. The accuracy of the method increases with the number of checking points, so it needs more factorization processes to get more exact results.

In this study, Gleyse’s theorem (Gleyse and Moflih 1999), which can count the number of zeros of a characteristic polynomial inside an open unit disk, is used to calculate the number of eigenvalues for eigenproblem with a damping matrix. The characteristic polynomial of an eigenvalue problem is determined by using Rombouts’ algorithm (Rombouts and Heyde 1998) which is considered as both stable and efficient
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algorithm. The determinants (minors) of the leading principal submatrices of order \( i \) in the Schur-Cohn matrix can be easily calculated by applying the \( \text{LDL}^T \) factorization process and the obtained final result is very similar to the Sturm sequence property for the undamped systems.
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1.2 Literature Survey

The well-known Sturm sequence property has hitherto been applied to check the missed eigenvalues (Meirovitch 1980; Hughes 1987; Petyt 1990; Bahte 1996). The technique for checking the missed eigenvalues using the Sturm sequence property is very important in that it is the only means to make sure that indeed the required number of eigenvalues has been evaluated. As discussed in the background of this study, this checking process is required to obtain the exact dynamic response. The technique using the Sturm sequence property is used in the commercial FEM programs such as ADINA. However, this technique can only be applied to the eigenproblem without damping matrix such as the case of the undamped and proportional damped system (Newland 1989).

In most real systems, the damping matrix is non-proportional (Caughey and O’Kelly 1965). Even when a proportional damping is assumed for each sub-system in the analysis of soil-structure interaction problem, structural control problem, composite structures, etc., the resulting damping matrix for the complete structure will be non-proportional. In these cases, the eigenvalue problem with the non-proportional damping should be analyzed for the exact dynamic response. By using the subspace iteration method (Olson and Vandini 1989; Leung 1995), the Lanczos method (Chen and Taylor 1988; Rajakumar 1993), the Arnoldi method (Arnoldi 1951; Ren and Zheng 1997) and the method developed by Lee et al.(1998) and Kim and Lee(1999), one can easily obtain the solution of this eigenproblem. However, although the eigenvalue analysis is
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performed well, the accuracy of the dynamic response is not guaranteed because the missed eigenvalues may exist in the required ones. To guarantee the exact dynamic response, a technique that can check the missed eigenvalues of structures with non-proportional damping (i.e., the eigenproblem including the damping matrix) is required.

In the field of the structural dynamics, the study on a technique for checking the missed eigenvalues in the eigenvalue analysis including the damping matrix has been rarely carried out up to now. On the other hand, in the field of the control system engineering, relevant researches have been carried out a few (Locher 1993; Tsai and Chen 1993; Yamada et al. 1998). In the above literature, for stability test of a system having complex zeros, the root distribution of the characteristic polynomial of the system was investigated. However, the techniques presented in the above literature are difficult to apply to an eigenproblem including damping matrix of a system, because all the techniques calculate zeros of a polynomial by using rigorous symbolic algebraic operations. Recently, Jung et al. (2001) proposed a technique of checking missed eigenvalues for eigenproblem with damping matrix using argument principle. This method requires a selection of checking points and the $LDL^T$ factorization of the characteristic polynomial at those points. The accuracy of the method increases with the number of checking points, so it needs more factorization processes to get more exact results.

As seen from the above review, a technique for checking the missed eigenvalues that is applicable to the eigenvalue analysis for the non-proportionally damped system has
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not been established yet. An appropriate technique for this case should be, therefore, developed to get the exact dynamic response as soon as possible.
1.3 Objectives and Scope

The purpose of this study is to develop an efficient numerical method for counting the number of eigenvalues inside an interested region for non-proportionally damped systems.

The objectives and scope of the study can be summarized as follows:

(1) Selection of an appropriate mathematical property to be applicable to a technique for checking the missed eigenvalues of structures with non-proportional damping:

(2) Development of a technique for checking the missed eigenvalues that can be applicable to the non-proportionally damped structures by using the above appropriate mathematical property:

Since an analytical solution by symbolic operations cannot be calculated, the numerical solution by complex arithmetic operations is considered. And, by analyzing the numerical examples, it is verified that the proposed method can check exactly the number of the missed eigenvalues.
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1.4 Organization

This dissertation consists of four chapters. Chapter 1 discusses the background, the literature review, and the objectives and scope of this study.

In Chapter 2, the argument principle based method by Jung et al. (2001) is reviewed. The conventional equations of motion of damped system are reviewed in Section 2.1. In Section 2.2, the argument principle for a characteristic polynomial is reviewed. The effects of discretization of the closed contour are explained in Section 2.3. The evaluation of the arguments for the characteristic polynomial along the closed contour is explained in Section 2.4. In Section 2.5, the evaluation procedure and considerations for argument principle based method are presented.

In Chapter 3, the modified Sturm sequence property for damped systems is proposed. Three algorithms for calculating the coefficients of characteristic polynomial are reviewed in Section 3.1. In Section 3.2, a theorem for calculating the number of eigenvalues in an open disk of unit radius by Gleyse and Moflih (1998) is explained. In Section 3.3, the modified Sturm sequence property for counting the number of eigenvalues in an open disk of arbitrary radius from the origin is proposed.

In Chapter 4, to verify the effectiveness of the proposed method, two numerical examples of a simple spring-mass-damper system and a plane frame structure with lumped dampers are considered.
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Finally, the conclusions of this study are summarized and some recommendations for the further studies are also presented in Chapter 5.
CHAPTER 2

REVIEW OF THE ARGUMENT PRINCIPLE-BASED METHOD

2.1 Equations of Motion of Damped Systems

In the analysis of dynamic response of structural systems, the equation of motion of damped systems can be written as:

\[ \mathbf{M} \ddot{u}(t) + \mathbf{C} \dot{u}(t) + \mathbf{K} u(t) = 0, \quad (2.1) \]

where \( \mathbf{M} \), \( \mathbf{C} \) and \( \mathbf{K} \) are the \( (n \times n) \) mass, non-proportional damping and stiffness matrices, respectively, and \( \ddot{u}(t) \), \( \dot{u}(t) \) and \( u(t) \) are the \( (n \times 1) \) acceleration, velocity and displacement vectors, respectively. To find the solution of the free vibration of the system, the following quadratic eigenproblem is considered:

\[ \lambda^2 \mathbf{M} \phi + \lambda \mathbf{C} \phi + \mathbf{K} \phi = 0, \quad (2.2) \]

in which \( \lambda \) and \( \phi \) are the eigenvalue and eigenvector of the system respectively. There are \( 2n \) eigenvalues for the system with \( n \) degrees of freedom and these occur either in real pairs or in complex conjugate pairs, depending upon whether they correspond to overdamped or underdamped modes.
In general, the mass matrix $\mathbf{M}$ is non-singular, that is $\det(\mathbf{M}) \neq 0$, and we can reformulate the quadratic system of equation to a state-space form by doubling the order of the system (Meirovitch 1990; Rajakumar 1993; Kim and Lee 1999) such as:

$$\mathbf{A}\psi = \lambda\psi,$$  \hspace{1cm} (2.3)

where

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \psi = \begin{bmatrix} \phi \\ \lambda\phi \end{bmatrix}. \hspace{1cm} (2.4)$$

The equation (2.3) is a standard eigenproblem, and the form of the matrix $\mathbf{A}$ in equation (2.4) is widely used in control engineering field (Meirovitch 1990).
2.2 Argument Principle for a Characteristic Polynomial (Jung et al. 2001)

Using the relationship between the eigenvalues of an eigenproblem and the zeros of the corresponding characteristic polynomial, the eigenvalues of the quadratic eigenproblem as equation (2.2) are equal to the zeros of the following characteristic polynomial:

\[
p(\lambda) = \det(\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K}) = a_{2n}\lambda^{2n} + a_{2n-1}\lambda^{2n-1} + \cdots + a_1\lambda + a_0, \quad (2.5)
\]

where \( \lambda \) is a complex value and \( a_k \) (\( k = 0,1,\ldots,2n \)) are real coefficients. The value of \( \lambda \) that satisfies \( p(\lambda) = 0 \) is called an eigenvalue of the system.

If the characteristic polynomial \( p(\lambda) \) is analytic inside and on a simple closed contour \( S \), the following argument principle can be applied (Franklin et al. 1998):

\[
N = \frac{1}{2\pi i} \oint_S \frac{p'(\lambda)}{p(\lambda)} d\lambda = \frac{\Delta \theta}{2\pi}, \quad (2.6)
\]

where \( N \) is the number of zeros of \( p(\lambda) \) inside the contour \( S \), \( \Delta \theta \) is the variation of the argument \( \theta \) of \( p(\lambda) \) around the contour \( S \) and \( i \) is defined as \( \sqrt{-1} \).

Because \( \Delta \theta/2\pi \) in the right side of equation (2.6) can be interpreted as the integer number of rotations, the characteristic polynomial \( p(\lambda) \) that maps a moving point \( \lambda \) describing the contour \( S \) into a moving point \( p(\lambda) \) encircles the origin of the \( p(\lambda) \)-plane \( N \) times if the polynomial \( p(\lambda) \) has \( N \) zeros inside the contour \( S \) in the \( \lambda \)-
Chapter 2. Review of the Argument Principle Based Method

plane. An example that the characteristic polynomial $p(\lambda)$ has two zeros inside a contour $S$ is shown in Figure 2.1. In Figure 2.1(a), the number of eigenvalues inside the contour $S$ is 2, and in Figure 2.1(b), the number of rotations of the characteristic polynomial $p(\lambda)$ about the origin is 2.

![Figure 2.1 Argument principle](image)

However, since it is difficult to directly evaluate equation (2.6) using the symbolic algebraic operations, the numerical, or the iterative, approach was developed to apply the aforementioned argument principle to the non-proportionally damped systems.

2.3 Discretization of the Contour $S$

The characteristic polynomial $p(\lambda)$ in equation (2.5) at point $\lambda = \lambda_j$ can be
factorized as follows:

\[ p(\lambda_j) = a_{2n} \prod_{k=1}^{2n} (\lambda_j - z_k) = r_j \angle \theta_j \]  \hspace{1cm} (2.7)

where \( z_k \) is the \( k \)-th zero of the \( p(\lambda) \), and \( r_j \) and \( \theta_j \) are the magnitude and argument of the value \( p(\lambda_j) \) in polar form, respectively.

If we consider \( \lambda_j \) and \( z_k \) as a vector on the complex plane, \( \lambda_j - z_k \) is also a vector from \( z_k \) to \( \lambda_j \). The polar form of \( \lambda_j - z_k \) can be written as:

\[ \lambda_j - z_k = r_{j,k} e^{i\theta_{j,k}} \]  \hspace{1cm} (2.8)

where \( r_{j,k} \) is the length and \( \theta_{j,k} \) is the argument of the \( \lambda_j - z_k \). A graphical representation for the polar form of \( \lambda_j - z_k \) is shown in Figure 2.2.

![Graphical representation of \( \lambda_j - z_k \)](image_url)

Using the polar form of \( \lambda_j - z_k \) as in equation (2.8), the \( p(\lambda_j) \) in equation (2.7) can be evaluated as:
Chapter 2. Review of the Argument Principle Based Method

\[ p(\lambda_j) = a_{2n} \prod_{k=1}^{2n} (\lambda_j - z_k) = a_{2n} r_{j,1} r_{j,2} \cdots r_{j,2n} e^{i(\theta_{j,1} + \theta_{j,2} + \cdots + \theta_{j,2n})} \]  (2.9)

So, \( r_j \) and \( \theta_j \) of the \( p(\lambda_j) \) in equation (2.7) have the following relationships.

\[ r_j = a_{2n} r_{j,1} r_{j,2} \cdots r_{j,2n} = a_{2n} \prod_{k=1}^{2n} r_{j,k} \]  (2.10)

\[ \theta_j = \theta_{j,1} + \theta_{j,2} + \cdots + \theta_{j,2n} = \sum_{k=1}^{2n} \theta_{j,k} \]  (2.11)

To consider the effect of the discretization of the contour \( S \), the values of \( p(\lambda) \) at two consecutive discrete points \( j=l \) and \( j=l+1 \) are evaluated as:

\[ p(\lambda_l) = a_{2n} \prod_{k=1}^{2n} (\lambda_l - z_k) = r_l \angle \theta_l \]  (2.12)

\[ p(\lambda_{l+1}) = a_{2n} \prod_{k=1}^{2n} (\lambda_{l+1} - z_k) = r_{l+1} \angle \theta_{l+1} \]  (2.13)

Then, the change of argument \( \Delta \theta_{l+1,l} \) from \( p(\lambda_l) \) to \( p(\lambda_{l+1}) \) can be obtained as:

\[ \Delta \theta_{l+1,l} = \theta_{l+1} - \theta_l = \sum_{k=1}^{2n} (\theta_{l+1,k} - \theta_{l,k}) \]  (2.14)

Since the argument change \( \Delta \theta_{l+1,l} \) in equation (2.14) is simply sum of the effects of each zero of the characteristic polynomial \( p(\lambda) \), we consider the simplest case that only one zero is inside the contour \( S \) as shown in Figure 2.3, where \( m \) is the number of discrete points on the closed contour \( S \).

The Figure 2.3(a) represents the case that a zero is inside the closed contour \( S \), and Figure 2.3(b) the case that a zero is outside the closed contour \( S \). When a zero is inside
Chapter 2. Review of the Argument Principle Based Method

the closed contour $S$, the sum of the argument change along discrete points on the $S$ is $2\pi$,
and when a zero is outside the closed contour $S$, the sum of the argument change is 0. So,
if we evaluate $p(\lambda)$ at some finite discrete point along the closed contour $S$, the sum of
the change of the argument of $p(\lambda)$ divided by $2\pi$ is exactly equal to the number of
zeros inside the contour $S$. From this figure we can also find that the argument change is
abrupt when the zero is close to the contour $S$.

2.4 Evaluation of the Arguments for $p(\lambda)$

The relationship between the characteristic polynomial and the factorized matrices
by the \textbf{LDL} factorization process can be used to evaluate the arguments of $p(\lambda)$. The
contour $S$ is considered as the set of the discrete checking points as shown in Figure 2.4.
Chapter 2. Review of the Argument Principle Based Method

Figure 2.4 Process of checking missed eigenvalues

The $LDL^T$ factorization process is performed at each checking point. Then, the argument at each checking point can be calculated as follows (Korn 1968; Pearson 1974):

$$p(\lambda_j) = \det(\lambda_j^2M + \lambda_jC + K) = \det LDL^T = \prod_{k=1}^{n} d_{kk} = r_j \angle \theta_j,$$  

(2.15)

where $d_{kk}$ are the diagonal elements of the diagonal matrix $D$, and $r_j$ and $\theta_j$ are the magnitude and argument of the value $p(\lambda_j)$ in polar form, respectively. The number of the eigenvalues inside the contour $S$ is calculated by summing the variations of the argument of each checking point in equation (2.14).

2.5 Evaluation Procedure and Considerations

In the implementation of the method for a practical problem, it is very important to properly choose the shape, the size and the number of discrete checking points of the
closed contour $S$. The simplest shape of the contour is a disk of given radius about the origin as shown in Figure 2.5(a). This shape can be applied to various damping cases such as underdamped, critically damped, and overdamped cases. Most of practical systems are underdamped and eigenvalues of the system are complex conjugates, so a half-circle and a line on the real axis as in Figure 2.5(b) are sufficient to check the missed eigenvalues in this case.

![Figure 2.5 Shapes of the contour $S$](image)

Because the argument change along the real line is 0, the number of checking points for the contour in Figure 2.5(b) is about half of that for the complete circle shown in Figure 2.5(a).

The size of the contour, i.e., the radius of a half-circle as in Figure 2.5(b) should be only a little bit larger than the largest eigenvalue to be considered to ensure that the next largest eigenvalue is not within the contour. The size of the contour recommended is
Chapter 2. Review of the Argument Principle Based Method

1.005 times the magnitude of the largest eigenvalue. This size is also used in the proposed method in the following section.

The number of checking points recommended is six times the number of considered eigenvalues \( r \). After the contour is equally divided into checking points, the part of the contour close to the largest eigenvalue is subdivided because the argument jump occurs in the part of the contour close to an eigenvalue. And, if the drastic change of the variation of the argument between two adjacent checking points occurs, the extra checking points between two adjacent checking points should be added. The detailed algorithm of the method is explained in Table 2.1.

The shape and size of the contour can be selected before applying the counting processes. However, the number of checking points varies during the processes when the drastic change of the variation of the argument occurs. And sometimes it is difficult to detect drastic change of the variation of the argument because the range of arguments is limited between \( 0^\circ \) to \( 360^\circ \) as well as it does not contain information about the number of rotations. If checking points are chosen sufficiently enough, the missed eigenvalues can exactly be checked by the method. However, the \( \text{LDL}^T \) factorizations of the characteristic polynomial at those points significantly increase the computational time.
Table 2.1 Algorithm of the argument principle based method

**Step 1: Calculate the maximum magnitude, $\rho$.**
- Select 1.005 times the absolute value of the $q$th eigenvalue ($\rho = 1.005|\lambda_q|$).

**Step 2: Determine the number and the location of the initial checking points.**
- **Number**: At least, select six times the number of the considered eigenvalues ($6r$).
- **Location**: Basically, divide into $6r$ equal parts.

**Step 3: Perform the checking process.**
1) Check the imaginary axis (the origin $\rightarrow \rho i$).
   - Perform the $LDL^T$ factorizing process at each checking point.
   - Calculate the argument $\theta_j$ at each checking point.

2) Check the second quarter-plane ($\rho \angle 90^\circ \rightarrow \rho \angle 180^\circ$).
   - Perform the $LDL^T$ factorizing process at each checking point.
   - Calculate the argument $\theta_j$ at each checking point.

**Step 4: Analyze the variations of the arguments.**
- If a decrease or aggressive variation of the argument occurs at a checking point, then go to Step 5 and if not, go to Step 6.

**Step 5: Add extra checking points.**
- Go to Step 3.

**Step 6: Check the missed eigenvalues and stop.**
- Calculate the total variation of the argument and the total rotation number.
- Compare the total rotation number ($N$ in equation (2.6)) with the number of the considered eigenvalues ($r$).
CHAPTER 3  

MODIFIED STURM SEQUENCE PROPERTY FOR DAMPED SYSTEMS  

3.1 The Coefficients of the Characteristic Polynomial  

The characteristic polynomial in equation (2.5) can be obtained another way using the matrix $A$ in equation (2.4) as:

$$p(\lambda) = \det(A - \lambda I) = \hat{a}_2 \lambda^2 + \hat{a}_1 \lambda + \hat{a}_0 = \sum_{k=0}^{2n} \hat{a}_k \lambda^k, \tag{3.1}$$

where $\lambda$ is a complex value and $\hat{a}_k (k = 0,1,\ldots,2n)$ are real coefficients. The coefficients $a_k (k = 0,1,\ldots,2n)$ in equation (2.5) are same scalar multiples to each $\hat{a}_k (k = 0,1,\ldots,2n)$ in equation (3.1).

There are several methods for calculating the coefficients of the characteristic polynomial of a real square matrix. The most famous one is Le Verrier’s algorithm (Faddeev and Faddeeva 1953), which is often described as a standard method in text books (Chen 1984; Franklin 1998). Wang and Chen (1982) pointed out the numerical instability and inefficiency of Faddeev-Le Verrier’s method, and proposed a numerically stable method to compute the characteristic polynomial based on Frobenius form of a matrix. This method needs to prescribe a small value to prevent some elements be divided by this and this value should be guided by error analysis and/or experience. Recently, Rombouts and Heyde (1998) presented an algorithm for calculating the
coefficients of the characteristic polynomial of a general square matrix for the evaluation of canonical traces in determinant quantum Monte-Carlo methods. This algorithm does not include division, so it is stable and also known as being efficient and accurate.

### 3.3.1 Le Verrier’s Algorithm

Le Verrier (1840) was the first to consider the coefficient of the characteristic polynomial of a given matrix by using Newton’s Formulae for symmetric functions. Except for some very short notes by Host (1935) and by Souriau (1948), nothing seems to have been added to what was known as Le Verrier’s algorithm until the late fifties of the present century. Recently, there appeared some papers using Le Verrier’s method, such as by Barnett’s (1989). So Le Verrier’s algorithm can be considered as the most widely used method for calculating the coefficients of the characteristic polynomial of a given matrix.

Each coefficient of the characteristic polynomial in equation (3.1) can be calculated as:

\[
\begin{align*}
\hat{a}_{2n} &= 1 \\
\hat{a}_{2n-1} &= -\frac{\text{tr}AR_0}{1}, R_0 = I \\
\hat{a}_{2n-2} &= -\frac{\text{tr}R_1}{2}, R_1 = AR_0 + \hat{a}_{2n}I = A + \hat{a}_{2n}I \\
& \vdots \\
\hat{a}_4 &= -\frac{\text{tr}AR_{2n-2}}{2n} \\
\hat{a}_3 &= \frac{\text{tr}R_{2n-2} + \hat{a}_4I}{2} = \hat{a}_4 + A^{2n-3}I \\
\hat{a}_2 &= \frac{-\text{tr}AR_{2n-3}}{2n} \\
\hat{a}_1 &= \frac{-\text{tr}R_{2n-3} + \hat{a}_2I}{2} = \hat{a}_2 + A^{2n-4}I \\
\hat{a}_0 &= \frac{-\text{tr}AR_{2n-4}}{2n} \\
0 &= AR_{2n-4} + \hat{a}_1I \\
\end{align*}
\]  

(3.2)
where $tr$ stands for the *trace* and is defined as the sum of all the diagonal elements of a matrix. If the order of a matrix is $k$, then this algorithm requires a total of $k$ times of multiplication of two $k \times k$ matrices. Hence the number of operations is proportional to $k^4$. Furthermore, this algorithm may encounter some numerical difficulties (Aplevich, 1974).

### 3.3.2 Chen’s Algorithm

Chen (1988) suggested a numerically stable algorithm to obtain the coefficients of the characteristic polynomial of a real square matrix.

A given matrix $A$:

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1,2n-1} & a_{1,2n} \\
  a_{21} & a_{22} & \cdots & a_{2,2n-1} & a_{2,2n} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{2n-1,1} & a_{2n-1,2} & \cdots & a_{2n-1,2n-1} & a_{2n-1,2n} \\
  a_{2n,1} & a_{2n,2} & \cdots & a_{2n,2n-1} & a_{2n,2n}
\end{bmatrix}, \quad (3.3)
\]

can be transformed to $\overline{A}$:

\[
\overline{A} = \begin{bmatrix}
  -a_{2n-1} & -a_{2n-2} & \cdots & -a_{1} & -a_{0} \\
  1 & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & 0 & 0 \\
  0 & 0 & \cdots & 1 & 0
\end{bmatrix}, \quad (3.4)
\]
by applying a sequence of Gauss-elimination like similarity transformations. Detailed algorithms are shown in Table 2.2. Since $\overline{A}$ was obtained by applying similar transformations to $A$, the eigenvalues and eigenvectors of both $\overline{A}$ and $A$ are same. The characteristic polynomial of $A$ can be obtained by observing the first row in the transformed matrix $\overline{A}$, and the characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I)$$

$$= \det(\overline{A} - \lambda I)$$

$$= \lambda^{2n} + a_{2n-1}\lambda^{2n-1} + \cdots + a_1\lambda + a_0 = 0, (a_{2n} = 1).$$

When some numerical instabilities due to very small off-diagonal elements occur during the transformations, the modified algorithm by Chen (1988) can be used.
Step 1: Choose two positive small numbers $\varepsilon_1$ and $\varepsilon_2$.
- $\varepsilon_1 < \varepsilon_2$.
- These two numbers are used to test small off-diagonal elements.

Step 2: Set $j=1$.

Step 3: Find $a_{ij}$, the largest in absolute value element of the set
\[ \{a_{ij}, i = j + 1, j + 2, \ldots, 2n\} \]
- If $r < j + 1$, go to step 4.

Step 4: Interchange the $(j+1)$th and the $r$th rows, then the $(j+1)$th and the $r$th columns.

Step 5: If $|a_{j+1,j}| \leq \varepsilon_1$, go to Step 6.
- Zero $a_{ij}(i = j + 2, j + 3, \ldots, 2n)$ by adding the product of the $(j+1)$th row with $(-a_{j+1,j})$ to the $i$th row.
- Add the product of the $i$th column with $(a_{j+1,j})$ to the $(j+1)$th column.

Step 6: If $|a_{j+1,j}| \leq \varepsilon_2$, go to Step 7.
- Multiply the $(j+1)$th row by $(1/a_{j+1,j})$.
- Multiply the $(j+1)$th column by $a_{j+1,j}$.

Step 7: If $j < 2n$, $j \rightarrow j + 1$, and go to Step 2.
- Set $i = 2n$.

Step 8: If $a_{i,j-1} \neq 1$, go to Step 9.
- Zero $a_{ij}(j = i, i + 1, \ldots, 2n)$ by adding the product of the $(i-1)$th column with $(-a_{ij})$ to the $j$th column.
- Add the product of the $j$th row with $a_{ij}$ to the $(j-1)$th row.

Step 9: If $i = 2$, go to Step 10.
- $i \rightarrow i - 1$ and go to Step 8.

Step 10: Stop.
- The matrix $A$ is transformed into the form $\overline{A}$.
3.3.3 Rombouts’ Algorithm

A general real square matrix $A$ as in equation (3.3) can be transformed to upper Hessenberg form $\hat{A}$:

$$\hat{A} = \begin{bmatrix}
\hat{a}_{1,1} & \hat{a}_{1,2} & \cdots & \hat{a}_{1,2n-1} & \hat{a}_{1,2n} \\
\hat{a}_{2,1} & \hat{a}_{2,2} & \cdots & \hat{a}_{2,2n-1} & \hat{a}_{2,2n} \\
0 & \hat{a}_{3,2} & \cdots & \hat{a}_{3,2n-1} & \hat{a}_{3,2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \hat{a}_{2n,2n-1} & \hat{a}_{2n,2n}
\end{bmatrix}, \quad (3.6)$$

by applying Householder reduction or sequence of Gaussian elimination like similarity transformations (Press et al. 1988).

Because the matrix $\hat{A}$ was obtained by applying similarity transformations to $A$, the eigenvalues of both $\hat{A}$ and $A$ are the same and the characteristic polynomials are different from each other by a scalar ratio to each other. For the purpose of calculating eigenvalues of the system, therefore, the characteristic polynomial can be considered as:

$$p(\lambda) = \det(\hat{A} - \lambda I). \quad (3.7)$$

If we define $\overline{p}(\lambda)$ as:

$$\overline{p}(\lambda) = \det(I + \lambda \hat{A}), \quad (3.8)$$

then this polynomial is closely related to $p(\lambda)$ as:
The basic idea of the Rombouts’ algorithm is to consider $\hat{A} + \lambda I$ as a matrix of polynomial in $\lambda$. We then calculate the polynomial $\overline{p}(\lambda)$ by evaluating the determinant in equation (3.8) using Gaussian elimination, with polynomials instead of scalars as matrix elements. As presented in equation (3.9) the coefficients of $\overline{p}(\lambda)$ are closely related to the coefficients of the $p(\lambda)$. The procedures of Rombouts’ algorithm for calculating the coefficients of the characteristic polynomial of a $2n$-by-$2n$ matrix $A$ are presented in Table 3.2.
Table 3.2 Rombouts’ algorithm for calculating characteristic polynomial

**Step 1: Reduce the given matrix** \( A \) (2n-by-2n) **to upper Hessenberg form** \( \hat{A} \).
- Use Householder reductions.

**Step 2: Initialize a matrix** \( B \) (2n-by-2n).
- Set all the elements of matrix \( B \) to 0.

**Step 3: Calculate element** \( b_{ij} \) **of the matrix** \( B \) **using the element** \( \hat{a}_{ij} \) **of the matrix** \( \hat{A} \) **as follows:**

\[
\text{DO } j = 2n, 1, -1 \\
\text{DO } i = 1, j \\
\text{DO } k = 2n-j, 1, -1 \\
\quad b_{k+1,j} = \hat{a}_{i,j} b_{k,j+1} - \hat{a}_{j+1,j} b_{k,j} \\
\text{ENDDO} \\
\quad b_{i,j} = \hat{a}_{i,j} \\
\text{ENDDO} \\
\text{DO } k = 1, 2n-j \\
\quad b_{k,j} = b_{k,j} + b_{k,j+1} \\
\text{ENDDO} \\
\text{ENDDO}
\]

**Step 4: Calculate the coefficients of the characteristic polynomial** \( \hat{a}_i \) \((i=0, \ldots, 2n)\) **and stop**
- Using the first row of the matrix \( B \), the coefficients of the characteristic polynomial can be computed as:

\[
\hat{a}_i = (-1)^{2n-i} b_{2n-i,1}
\]
3.2 The Number of Eigenvalues in an Unit Open Disk

Gleyse and Moflih (1999) suggested a method of calculating the number of eigenvalues of a real polynomial in a unit open circle as shown in Figure 3.1 by a determinant representation.

Let $k_n$ be the number of eigenvalues in a unit open circle, $n$ is the degree of the polynomial, $\nu[1, d_1, d_2, \cdots, d_{2n}]$ is the number of sign changes in the unit disk

$$N_o = 2n - \nu[1, d_1, d_2, \cdots, d_{2n}]$$  

where $N_o$ is the number of eigenvalues in a unit open circle, $2n$ is the degree of the polynomial, $\nu[d_0, d_1, d_2, \cdots, d_{2n}]$ is the number of sign changes in the
sequence \( d_i \) \( (i = 0, 1, \cdots, 2n) \) and \( d_i \) \( (i = 1, 2, \cdots, 2n) \) are the determinants (minors) of the leading principal submatrices of order \( i \) in the Schur-Cohn matrix \( T \):

\[
T = \begin{bmatrix}
T_1 & \cdots & T_i & \cdots & T_{2n} \\
T_{i1} & \cdots & t_{ii} & \cdots & t_{i,2n} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
t_{2n,1} & \cdots & t_{2n,i} & \cdots & t_{2n,2n}
\end{bmatrix}
\]

\[
t_{ij} = \sum_{h=0}^{\min(i,j)} (\hat{a}_{2n-i+h} \hat{a}_{2n-j+h} - \hat{a}_{i-h} \hat{a}_{j-h}) \quad (i, j = 1, 2, \cdots, 2n) \quad (3.11)
\]

\[
d_i = \det(T_i) \quad (3.12)
\]

The processes of counting the number of eigenvalues inside a unit open circle by the above method require calculation of the characteristic polynomial of a given matrix \( A \), the construction of the Schur-Cohn matrix \( T \) and the calculation of the determinants (minors) of the leading principal submatrices of order \( i \) in the Schur-Cohn matrix \( T \). The coefficients of the characteristic polynomial of a given matrix can be determined by Rombouts’ algorithm described in the previous section 3.1.3, and each element of the Schur-Cohn matrix can be obtained using equation (3.11).
3.3 Proposed Method

Gleyse’s theorem (Gleyse and Moflih 1999) considers only the number of eigenvalues in a unit open circle. To apply this theorem to an open circle of arbitrary radius $\rho$, we substitute $\lambda = \rho \bar{\lambda}$ ($\rho$ is a real number) to equation (3.1), then the modified characteristic polynomial can be written as:

$$P(\bar{\lambda}) = \tilde{a}_n \rho^{2n} \bar{\lambda}^{2n} + \tilde{a}_{2n-1} \rho^{2n-1} \bar{\lambda}^{2n-1} + \cdots + \tilde{a}_1 \rho \bar{\lambda} + \tilde{a}_0$$

$$= \tilde{a}_2 \rho^{2n} + \tilde{a}_{2n-1} \rho^{2n-1} + \cdots + \tilde{a}_1 \rho + \tilde{a} = \sum_{k=0}^{2n} \tilde{a}_k \rho^k,$$

where $\tilde{a}_k = \hat{a}_k \rho^k (k = 0, 1, \cdots, 2n)$ are modified coefficients. Using the modified coefficients $\tilde{a}_k (k = 0, 1, \cdots, 2n)$ in equation (3.13), this theorem can be extended to calculate the number of eigenvalues in the open disk of arbitrary radius $\rho$ as shown in Figure 3.2.

![Figure 3.2 Eigenvalues in an open disk of radius $\rho$](image)

(○: eigenvalues outside the open disk, ●: eigenvalues in the open disk)
The calculation of \( d_i (i = 1, \ldots, 2n) \) can be easily performed by the \( \text{LDL}^T \) factorization of the Schur-Cohn matrix \( T \). If \( T = \text{LDL}^T \), then:

\[
\textbf{T}_j = \textbf{L}_j \textbf{D}_j \textbf{L}_j^T ,
\]

where the matrix \( \textbf{T}_j \) is the leading principal submatrices of order \( i \) in the Schur-Cohn matrix \( T \), the matrix \( \textbf{L}_j \) is the leading principal submatrices of order \( i \) in the factorized lower triangular matrix \( \textbf{L} \), and the matrix \( \textbf{D}_j \) is the leading principal submatrices of order \( i \) in the factorized diagonal matrix \( \textbf{D} \) as shown in equation (3.15). The values of \( d_i (i = 1, \ldots, 2n) \) can be evaluated as:

\[
d_i = \det(\textbf{T}_i) = \det(\textbf{L}_i \textbf{D}_i \textbf{L}_i^T) = \det(\textbf{D}_i) = \prod_{h=1}^{i} d_{hh}.
\]

Therefore, each \( d_i = \det(\textbf{T}_i) \) can be obtained by multiplying from the first diagonal element \( d_{i1} \) to the \( i \)-th diagonal element \( d_{ii} \) of the factorized diagonal matrix \( \textbf{D} \).
Chapter 3 Modified Sturm Sequence Property for Damped Systems

Considering equation (3.10), we only need to know the sign of each $d_i$ because the unknown value of $V[l, d_1, d_2, \ldots, d_{2n}]$ depends on sign changes of each $d_i \ (i = 1, \ldots, 2n)$, and from equation (3.16) the sign change of $d_i$ from $d_{i-1}$ occurs when the diagonal element $d_{ii}$ of the factorized diagonal matrix $D$ is negative. So, the value of $V[l, d_1, d_2, \ldots, d_{2n}]$ is equal to the number of negative element in the matrix $D$ which can be proven by mathematical induction. If we combine this result with equation (3.10), the number of eigenvalues inside an open disk of radius $\rho$ and the number of positive elements the factorized diagonal matrix $D$ have the following relationship:

$$N_\rho = \text{the number of positive elements in the diagonal matrix } D \quad (3.17)$$

where $N_\rho$ is the number of eigenvalues inside an open disk of radius $\rho$ and $D$ is the diagonal matrix obtained by factorization of Schur-Cohn matrix $T$ constructed using the modified coefficients in equation (3.13). This relation is very similar to the Sturm sequence property for undamped systems. The algorithm of the proposed method is summarized in Table 3.3.
Table 3.3  Algorithm of the proposed method

\textbf{Step 1: Change to a standard eigenproblem.}
\begin{itemize}
  \item Change the given eigenproblem to a standard form.
\end{itemize}

\textbf{Step 2: Calculate the coefficients of the characteristic polynomial.}
\begin{itemize}
  \item Using Chen’s algorithm, construct \( p(\lambda) = \sum_{k=0}^{2n} a_k \lambda^k = 0 \).
\end{itemize}

\textbf{Step 3: Determine the radius \( \rho \) of an open disk.}
\begin{itemize}
  \item The radius \( \rho \) can be arbitrary, but select a little bit larger magnitude than the largest known eigenvalue to check missed eigenvalues.
\end{itemize}

\textbf{Step 4: Modify the coefficients of the characteristic polynomial.}
\begin{itemize}
  \item Substitute \( \lambda = \rho \bar{\lambda} \) to \( p(\lambda) \) obtained at Step 2.
\end{itemize}

\textbf{Step 5: Construct the Schur-Cohn matrix.}
\begin{itemize}
  \item Construct the Schur-Cohn matrix \( T \) using the modified coefficients of the characteristic polynomial.
\end{itemize}

\textbf{Step 6: Perform the LDL\textsuperscript{T} factorization the Schur-Cohn matrix \( T \).}

\textbf{Step 7: Calculate (check) the number eigenvalues inside the open disk and stop.}
\begin{itemize}
  \item Calculate the number of positive elements in the diagonal of \( D \).
  \item The number of eigenvalues is equal to the number of positive elements in \( D \).
\end{itemize}
To show the applicability of the proposed method, two numerical examples are analyzed. A simple spring-mass-damper system having the exact analytical eigenvalues is considered to verify that the method can exactly calculate the number of eigenvalues in the open disk of arbitrary radius for the eigenproblem with the damping matrix. The plane frame structure with lumped dampers is also considered to verify the method for the system with multiple eigenvalues.

4.1 Simple Spring-Mass-Damper System (Chen 1993)

A simple spring-mass-damper system is shown in Figure 4.1.

The finite element discretization of the system results in a diagonal mass matrix, a
Chapter 4 Numerical Examples

tridiagonal damping and stiffness matrices of the following forms:

\[ M = mI \]  
\[ C = \alpha M + \beta K \]  
\[ K = k \begin{bmatrix} 2 & -1 & \cdot & \cdot & \cdot & -1 \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \]  

The analytical solutions can be resulted through following relationships:

\[ \lambda_{2j-1,2j} = -\xi_j \omega_j \pm i \omega_j \sqrt{1 - \xi_j^2} \quad \text{for } j = 1, \ldots, n \]  
\[ \xi_j = \frac{1}{2} \left( \frac{\alpha}{\omega_j} + \beta \omega_j \right) \]  
\[ \omega_j = 2 \frac{m}{k} \sin \frac{2j-1}{2n+1} \pi \]  

where \( \omega_j \) and \( \xi_j \) are the undamped natural frequency and modal damping ratio, respectively and \( i \) is defined as \( \sqrt{-1} \). A system with order 10 is used in analysis. All the eigenvalues and their radii from the origin in the complex plane are shown in Table 4.1.

The number of considered eigenvalues is 8 and the radius of the contour \( S \) of the open circle is calculated by the 1.005 times the magnitude of the eighth eigenvalue (\( \rho = \)
Chapter 4 Numerical Examples

1.005 | \lambda_s | = 1.005). The half-circle of the contour S is initially divided into 24 (r=4) equal points as shown in Figure 4.2.

Table 4.1 Calculated eigenvalues of example 4.1

| Mode Number | Eigenvalues(\lambda) | Radius (\rho = |\lambda|) |
|-------------|----------------------|--------------------------|
|             | Real                 | Imaginary                |
| 1, 2        | -0.0306              | ±0.1463                  | 0.1495 |
| 3, 4        | -0.0745              | ±0.4388                  | 0.1495 |
| 5, 6        | -0.1585              | ±0.7133                  | 0.4450 |
| 7, 8        | -0.2750              | ±0.9614                  | 1.0000 |
| 9, 10       | -0.4137              | ±1.1763                  | 1.2470 |
| 11, 12      | -0.5624              | ±1.3540                  | 1.4661 |
| 13, 14      | -0.7077              | ±1.4932                  | 1.6525 |
| 15, 16      | -0.8368              | ±1.5959                  | 1.8019 |
| 17, 18      | -0.9381              | ±1.6651                  | 1.9111 |
| 19, 20      | -1.0028              | ±1.7046                  | 1.9777 |

Figure 4.2 Contour of S of example 4.1
Chapter 4  Numerical Examples

Since the argument of the largest eigenvalue is 105.96°, the part of the contour between 101.74° and 109.57° is subdivided into six equal parts. The total variation of the argument is 1440° as in Table 4.2, and the number of rotations is

\[
N = \frac{\sum_{j=1}^{2n-1} \Delta \theta_j}{2\pi} = \frac{1440^\circ}{360^\circ} = 4
\]  \hspace{1cm} (4.7)

So, the number of eigenvalues inside a circle of radius \( \rho \) (\( = 1.005 \lambda_8 \)) is 8 (\( =4\times2 \)), which exactly agrees with the calculated value in Table 4.1. If we assume that the largest eigenvalue is unknown and use the same radius for the contour \( S \), we can detect the checking point where the drastic change of the variation of the argument occurs in this case as shown in Table 4.2. In Table 4.2, the change of the variation of the argument is 187.684° at checking point \( \rho \angle 109.57^\circ \). Because this value is over 180°, we can conclude that new checking points are needed. The contour of \( p(S) \) is shown in Figure 4.3. It is very difficult to count the number of rotations from Figure 4.3.

Because only the arguments are important to count the number of eigenvalues, the magnitude of \( p(S) \) can be scaled to help the graphical interpretation as shown in Figure 4.4. The number of rotations for the contour \( p(S) \) in the figure is 4.

The results for the modified Sturm sequence property are shown in Table 4.3. As shown at the last column in Table 4.3, the number of sign changes is 12. So if we use the equation (3.10), the number of eigenvalues inside the circle is 20-12 =8.
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Table 4.2 Arguments and the variations of the arguments of example 4.1

<table>
<thead>
<tr>
<th>First checking process</th>
<th>Second checking process</th>
<th>∑ ∆θj</th>
</tr>
</thead>
<tbody>
<tr>
<td>zj</td>
<td>θj</td>
<td>Δθj</td>
</tr>
<tr>
<td>origin</td>
<td>0.00</td>
<td>-</td>
</tr>
<tr>
<td>θ &lt; 7.83°</td>
<td>76.70</td>
<td>76.70</td>
</tr>
<tr>
<td>7.83° ≤ θ &lt; 15.65°</td>
<td>153.30</td>
<td>76.60</td>
</tr>
<tr>
<td>15.65° ≤ θ &lt; 23.48°</td>
<td>229.69</td>
<td>76.39</td>
</tr>
<tr>
<td>23.48° ≤ θ &lt; 31.30°</td>
<td>305.74</td>
<td>76.05</td>
</tr>
<tr>
<td>31.30° ≤ θ &lt; 39.13°</td>
<td>381.29</td>
<td>75.55</td>
</tr>
<tr>
<td>39.13° ≤ θ &lt; 46.96°</td>
<td>456.11</td>
<td>74.82</td>
</tr>
<tr>
<td>46.96° ≤ θ &lt; 54.78°</td>
<td>529.91</td>
<td>73.80</td>
</tr>
<tr>
<td>54.78° ≤ θ &lt; 62.61°</td>
<td>602.24</td>
<td>72.34</td>
</tr>
<tr>
<td>62.61° ≤ θ &lt; 70.43°</td>
<td>672.50</td>
<td>70.26</td>
</tr>
<tr>
<td>70.43° ≤ θ &lt; 78.26°</td>
<td>739.75</td>
<td>67.25</td>
</tr>
<tr>
<td>78.26° ≤ θ &lt; 86.09°</td>
<td>802.58</td>
<td>62.82</td>
</tr>
<tr>
<td>86.09° ≤ θ &lt; 93.91°</td>
<td>858.32</td>
<td>55.75</td>
</tr>
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<td>43.22</td>
</tr>
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<td>101.74° ≤ θ &lt; 109.57°</td>
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<td>38.64</td>
</tr>
<tr>
<td>109.57° ≤ θ &lt; 117.39°</td>
<td>1108.0</td>
<td>34.82</td>
</tr>
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<td>117.39° ≤ θ &lt; 125.22°</td>
<td>1132.4</td>
<td>31.00</td>
</tr>
<tr>
<td>125.22° ≤ θ &lt; 133.04°</td>
<td>1168.1</td>
<td>27.20</td>
</tr>
<tr>
<td>133.04° ≤ θ &lt; 140.87°</td>
<td>1212.9</td>
<td>23.40</td>
</tr>
<tr>
<td>140.87° ≤ θ &lt; 148.70°</td>
<td>1264.4</td>
<td>19.60</td>
</tr>
<tr>
<td>148.70° ≤ θ &lt; 156.52°</td>
<td>1320.6</td>
<td>15.80</td>
</tr>
<tr>
<td>156.52° ≤ θ &lt; 164.35°</td>
<td>1379.6</td>
<td>12.00</td>
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<td>164.35° ≤ θ &lt; 172.17°</td>
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<td>8.20</td>
</tr>
<tr>
<td>172.17° ≤ θ &lt; 180.00°</td>
<td>1500.0</td>
<td>4.40</td>
</tr>
</tbody>
</table>

where 0° ≤ θj < 360°, ∆θj = θj - θj-1, and 'Y' means that the additional checking points are required and 'N' means that the additional ones are not required.
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Figure 4.3 Contour of \( p(S) \)

(a) \( p(S) \) at 6\( r(=24) \) points        (b) \( p(S) \) at 1000 points

Figure 4.4 Scaled contour of \( p(S) \) S of example 4.1

(a) \( p(S) \) at 6 \( r(=24) \) points        (b) \( p(S) \) at 6\( r+6 \) points        (c) \( p(S) \) at 1000 points

Using equation (3.17), the number of positive elements in the matrix \( \mathbf{D} \) is 8 as shown at the third column in the Table 4.3.
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Table 4.3 Coefficients $\bar{a}_i$, diagonal elements $d_{ii}$ of $D$, and signs of $d_i$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\bar{a}_i$</th>
<th>$d_{ii}$</th>
<th>sign of $d_i$</th>
<th>$V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000e+000</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>1.4500e-001</td>
<td>2.2079e-001</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>1.9009e+001</td>
<td>3.6950e-002</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>2.3853e+000</td>
<td>-4.4940e+004</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1.5313e+002</td>
<td>-7.4052e+003</td>
<td>+</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>1.6424e+001</td>
<td>-2.0035e+003</td>
<td>-</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>6.8074e+002</td>
<td>-2.0029e+003</td>
<td>+</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>6.0218e+001</td>
<td>-1.1531e+003</td>
<td>-</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>1.8222e+003</td>
<td>-1.1531e+003</td>
<td>+</td>
<td>6</td>
</tr>
<tr>
<td>9</td>
<td>1.2972e+002</td>
<td>-2.7488e+002</td>
<td>-</td>
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</tr>
<tr>
<td>10</td>
<td>3.0067e+003</td>
<td>-2.7473e+002</td>
<td>+</td>
<td>8</td>
</tr>
<tr>
<td>11</td>
<td>1.6522e+002</td>
<td>4.5463e+002</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>12</td>
<td>3.0065e+003</td>
<td>4.5454e+002</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>13</td>
<td>1.2015e+002</td>
<td>-1.0560e+003</td>
<td>-</td>
<td>9</td>
</tr>
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<td>14</td>
<td>1.7177e+003</td>
<td>-1.0556e+003</td>
<td>+</td>
<td>10</td>
</tr>
<tr>
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<td>4.5550e+001</td>
<td>2.0436e+001</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>16</td>
<td>4.9538e+002</td>
<td>2.0433e+001</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>17</td>
<td>7.4260e+000</td>
<td>-6.6555e+000</td>
<td>-</td>
<td>11</td>
</tr>
<tr>
<td>18</td>
<td>5.5026e+001</td>
<td>-6.6555e+000</td>
<td>+</td>
<td>12</td>
</tr>
<tr>
<td>19</td>
<td>3.2500e-001</td>
<td>7.1550e-001</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>20</td>
<td>1.0000e+000</td>
<td>7.1549e-001</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

where $V$ represents the number of sign changes of $d_i$, ‘+’ means positive value and ‘−’ means negative value. The $d_0$ is defined as 1.
4.2 Plane Frame Structure with Lumped Dampers (Kim and Lee 1999)

In this example, a plane frame structure with lumped dampers is presented. The geometric configuration and material properties are shown in Figure 4.5.

![Diagram of Plane Frame Structure with Lumped Dampers]

*Figure 4.5 Plane frame structure with lumped dampers*

Young’s Modulus: 1000  \[ \text{Mass Density: 1.0} \]
Cross-section Inertia: 1.0  \[ \text{Cross-section Area: 1.0} \]
Span Length : L= 6.0  \[ \text{Concentrated Damping: 0.3} \]
Rayleigh Damping Coeff.: \[ \alpha=0.001, \beta=0.001 \]

The model is discretized in 6 beam elements with equal length for each direction resulting in the system of dynamic equation with a total of 18 degrees of freedom. Thus, the order of the associated eigenproblem is 36. The consistent damping matrix is derived from the classical damping given by \[ \mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K} \] and concentrated dampers resulting...
Chapter 4 Numerical Examples

in non-proportional damping matrix. All eigenvalues are calculated by the modified Lanczos method developed by Kim and Lee (1999) and their radii from the origin in the complex plane are calculated by \( \rho = |\lambda| \) as in Table 4.4.

| Mode Number | Eigenvalues(\( \lambda \)) | Radius \( (\rho = |\lambda|) \) |
|-------------|--------------------------|--------------------------|
| 1,2         | -1.1369 ±46.2187         | 46.2327                  |
| 3,4         | -1.1369 ±46.2187         | 46.2327                  |
| 5,6         | -1.3731 ±51.1333         | 51.1517                  |
| 7,8         | -1.3731 ±51.1333         | 51.1517                  |
| 9,10        | -3.3902 ±81.0872         | 81.1490                  |
| 11,12       | -3.3902 ±81.0872         | 81.1490                  |
| 13,14       | -3.9407 ±87.4771         | 87.5659                  |
| 15,16       | -3.9407 ±87.4771         | 87.5659                  |
| 17,18       | -8.1642 ±127.4394        | 127.7006                 |
| 19,20       | -8.1642 ±127.4394        | 127.7006                 |
| 21,22       | -10.2629 ±142.8367       | 143.2049                 |
| 23,24       | -10.2629 ±142.8367       | 143.2049                 |
| 25,26       | -14.8662 ±171.7301       | 172.3720                 |
| 27,28       | -14.8662 ±171.7301       | 172.3720                 |
| 29,30       | -20.5387 ±201.6249       | 202.6683                 |
| 31,32       | -20.5387 ±201.6249       | 202.6683                 |
| 33,34       | -23.7699 ±216.7332       | 218.0328                 |
| 35,36       | -23.7699 ±216.7332       | 218.0328                 |

The number of considered eigenvalues is 8 and the radius of the contour \( S \) is calculated by the 1.005 times the magnitude of the eighth eigenvalue \( (\rho = 1.005|\lambda_8| = 51.4075) \). The half-circle of the contour \( S \) is initially divided into 24 equal points. And
Chapter 4 Numerical Examples

since the argument of the largest eigenvalue is 91.54°, the part of the contour between
86.09° and 93.91° is subdivided into seven equal parts as shown in Figure 4.6.

![Figure 4.6 Contour of S of example 4.2](image)

Since the total variation of the argument is 1440° as in Table 4.5, the number of
rotations is

\[
N = \frac{\sum_{j=1}^{2n-1} \Delta \theta_j}{2\pi} = \frac{1440^\circ}{360^\circ} = 4.
\]  

(4.8)

So, the number of eigenvalues inside a circle of radius \( \rho \) \( = 1.005|\lambda_8| \) is 8 (\( = 4 \times 2 \)),
which exactly agrees with that obtained in Table 4.4. If we assume that the largest
eigenvalue is unknown and use the same radius for the contour \( S \), it is very difficult to
detect the checking point where the drastic change of the variation of the argument occurs
in this case. In Table 4.5, the change of the variation of the argument is 85.44° at
checking point \( \rho \angle 93.91^\circ \).
Table 4.5 Arguments and the variations of the arguments of example 4.2

<table>
<thead>
<tr>
<th>z_j</th>
<th>θ_j</th>
<th>Δθ_j</th>
<th>Y/N</th>
<th>θ_j</th>
<th>Δθ_j</th>
<th>Y/N</th>
<th>ΣΔθ_j</th>
</tr>
</thead>
<tbody>
<tr>
<td>origin</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;7.83°</td>
<td>67.26</td>
<td>67.26</td>
<td>N</td>
<td>67.26</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;15.65°</td>
<td>133.49</td>
<td>66.23</td>
<td>N</td>
<td>133.48</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;23.48°</td>
<td>197.60</td>
<td>64.12</td>
<td>N</td>
<td>197.60</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;31.30°</td>
<td>258.47</td>
<td>60.86</td>
<td>N</td>
<td>258.47</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;39.13°</td>
<td>314.83</td>
<td>56.36</td>
<td>N</td>
<td>314.83</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;46.96°</td>
<td>408.47</td>
<td>50.49</td>
<td>N</td>
<td>408.47</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;54.78°</td>
<td>468.09</td>
<td>43.15</td>
<td>N</td>
<td>468.09</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;62.61°</td>
<td>526.22</td>
<td>38.46</td>
<td>N</td>
<td>526.22</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;70.43°</td>
<td>584.44 or -274.56</td>
<td>-</td>
<td>-</td>
<td>584.44 or -274.56</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;78.26°</td>
<td>642.89</td>
<td>34.42</td>
<td>N</td>
<td>642.89</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;86.09°</td>
<td>701.20</td>
<td>25.20</td>
<td>N</td>
<td>701.20</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;93.91°</td>
<td>759.61</td>
<td>19.66</td>
<td>N</td>
<td>759.61</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;101.74°</td>
<td>818.02</td>
<td>14.54</td>
<td>N</td>
<td>818.02</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;109.57°</td>
<td>876.44</td>
<td>14.52</td>
<td>N</td>
<td>876.44</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;117.39°</td>
<td>934.86</td>
<td>19.08</td>
<td>N</td>
<td>934.86</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;125.22°</td>
<td>993.28</td>
<td>30.75</td>
<td>N</td>
<td>993.28</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;133.04°</td>
<td>1051.70</td>
<td>34.42</td>
<td>N</td>
<td>1051.70</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;140.87°</td>
<td>1109.12</td>
<td>25.20</td>
<td>N</td>
<td>1109.12</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;148.70°</td>
<td>1167.54</td>
<td>19.66</td>
<td>N</td>
<td>1167.54</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;156.52°</td>
<td>1225.96</td>
<td>14.54</td>
<td>N</td>
<td>1225.96</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;164.35°</td>
<td>1284.38</td>
<td>14.52</td>
<td>N</td>
<td>1284.38</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;172.17°</td>
<td>1342.80</td>
<td>19.08</td>
<td>N</td>
<td>1342.80</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
<tr>
<td>ρ&lt;180.00°</td>
<td>1401.22</td>
<td>30.75</td>
<td>N</td>
<td>1401.22</td>
<td>-</td>
<td>-</td>
<td>0.00</td>
</tr>
</tbody>
</table>

where 0° ≤ θ_j < 360°, Δθ_j = θ_j - θ_{j-1}, and 'Y' means that the additional checking points are required and 'N' the additional ones are not required.
Because this value is smaller than $180^\circ$, we cannot conclude whether new checking points are needed or not. The scaled contour of $p(S)$ is shown in Figure 4.7. The number of rotations for the contour $p(S)$ in the figure is 4.

![Scaled contour of $p(S)$](image)

(a) $p(S)$ at 6$r$+6($=30$) points  
(b) $p(S)$ at 1000 points

Figure 4.7 Scaled contour of $p(S)$

The results of the modified Sturm sequence property are shown in Table 4.6. As shown at the last column in Table 4.6, the number of sign changes is 28. So if we use the equation (3.10), the number of eigenvalues inside the circle is 36-28 =8. Using equation (3.17), the number of positive elements in the matrix $D$ is 8 as shown at the third column in the Table 4.6. Therefore, we verify that the method can exactly check the number of eigenvalues inside some open disk of arbitrary radius for the system with multiple eigenvalues.
Table 4.6 Coefficients $\bar{a}_i$, diagonal elements $d_{ii}$ of $D$, and signs of $d_i$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\bar{a}_i$</th>
<th>$d_{ii}$</th>
<th>sign of $d_i$</th>
<th>$V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.5017e+006</td>
<td>-</td>
<td>+</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2.3595e+006</td>
<td>-6.2584e+006</td>
<td>_</td>
<td>1</td>
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<tr>
<td>2</td>
<td>1.8523e+007</td>
<td>-6.2584e+006</td>
<td>+</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1.5836e+007</td>
<td>-6.2584e+006</td>
<td>_</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>6.0360e+007</td>
<td>-6.2584e+006</td>
<td>+</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>4.6688e+007</td>
<td>-6.2584e+006</td>
<td>_</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>1.1462e+008</td>
<td>-6.2584e+006</td>
<td>+</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>8.0016e+007</td>
<td>-6.2584e+006</td>
<td>_</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>1.4176e+008</td>
<td>-6.2584e+006</td>
<td>+</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>8.9075e+007</td>
<td>-6.2584e+006</td>
<td>_</td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>1.2111e+008</td>
<td>-6.2584e+006</td>
<td>+</td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>6.8279e+007</td>
<td>-6.2584e+006</td>
<td>_</td>
<td>11</td>
</tr>
<tr>
<td>12</td>
<td>7.4043e+007</td>
<td>-6.2584e+006</td>
<td>+</td>
<td>12</td>
</tr>
<tr>
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<td>3.7327e+007</td>
<td>-6.2584e+006</td>
<td>_</td>
<td>13</td>
</tr>
<tr>
<td>14</td>
<td>3.3137e+007</td>
<td>-6.2583e+006</td>
<td>+</td>
<td>14</td>
</tr>
<tr>
<td>15</td>
<td>1.4882e+007</td>
<td>-6.2576e+006</td>
<td>_</td>
<td>15</td>
</tr>
<tr>
<td>16</td>
<td>1.1017e+007</td>
<td>-6.2563e+006</td>
<td>+</td>
<td>16</td>
</tr>
<tr>
<td>17</td>
<td>4.3882e+006</td>
<td>-6.2388e+006</td>
<td>_</td>
<td>17</td>
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<tr>
<td>18</td>
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<td>-6.2202e+006</td>
<td>+</td>
<td>18</td>
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<tr>
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<td>-6.0086e+006</td>
<td>_</td>
<td>19</td>
</tr>
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<td>-5.9203e+006</td>
<td>+</td>
<td>20</td>
</tr>
<tr>
<td>21</td>
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<td>-4.8338e+006</td>
<td>_</td>
<td>21</td>
</tr>
<tr>
<td>22</td>
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<td>-4.7180e+006</td>
<td>+</td>
<td>22</td>
</tr>
<tr>
<td>23</td>
<td>1.9357e+004</td>
<td>-2.4454e+006</td>
<td>_</td>
<td>23</td>
</tr>
<tr>
<td>24</td>
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<td>-2.4029e+006</td>
<td>+</td>
<td>24</td>
</tr>
<tr>
<td>25</td>
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<td>-5.4284e+005</td>
<td>_</td>
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</tr>
<tr>
<td>26</td>
<td>5.9414e+002</td>
<td>-5.3645e+005</td>
<td>+</td>
<td>26</td>
</tr>
<tr>
<td>27</td>
<td>1.1381e+002</td>
<td>-6.1754e+003</td>
<td>_</td>
<td>27</td>
</tr>
<tr>
<td>28</td>
<td>3.3379e+001</td>
<td>-4.1125e+003</td>
<td>+</td>
<td>28</td>
</tr>
<tr>
<td>29</td>
<td>5.1996e+000</td>
<td>2.4923e+004</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>30</td>
<td>1.3074e+000</td>
<td>1.3928e+004</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>31</td>
<td>1.5712e-001</td>
<td>3.0169e+003</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>3.3582e-002</td>
<td>2.8778e+003</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>33</td>
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<td>2.5915e+000</td>
<td>+</td>
<td>-</td>
</tr>
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<td>34</td>
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<td>2.5924e+000</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>35</td>
<td>2.2413e-005</td>
<td>2.8473e-004</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>36</td>
<td>3.2943e-006</td>
<td>1.6343e-004</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

where $V$ represents the number of sign changes of $d_i$, ‘+’ means positive value and ‘−’ means negative value. The $d_0$ is defined as 1.
Methods of counting the number of eigenvalues for non-proportionally damped systems have been presented. The method based on argument principle requires many factorization processes at many checking points and sometimes it is difficult to detect drastic change of the variation of the argument. On the other hand, the proposed method needs only one factorization of the Schur-Cohn matrix. The final checking of the method is done by counting positive diagonal elements of the factorized Schur-Cohn matrix, which is very similar to the well known Sturm sequence property. As demonstrated in two numerical examples, the proposed method can exactly calculate the number of eigenvalues in an open disk of given radius.

The proposed method is based on well-proven algorithms and theorems, but during calculation of the coefficients of the characteristic polynomial, some small numerical errors may be accumulated mainly due to memory limitation. To apply the proposed method to large structures, therefore, further research to reduce the effects of numerical errors should be performed.
List of References

LIST OF REFERENCES


List of References


List of References


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