Chapter 4 Response of SDOF Systems to Harmonic Excitation

§ 4.1 Response of Undamped SDOF System to Harmonic Excitation

\[ m\ddot{u} + ku = p_o \cos \Omega t \quad (4.1) \]

Let

\[ u_p = U \cos \Omega t \quad \text{steady-state response} \quad (4.2) \]

Then

\[ U = \frac{p_o}{k-m\Omega^2} = \frac{p_o/\Omega^2}{k \left( 1 - \frac{\Omega^2}{\omega_n^2} \right)} \quad \text{amplitude of } u_p \quad (4.3) \]

Let

\[ U_o = \frac{p_o}{k} \quad \text{static displacement} \quad (4.4) \]

\[ H(\Omega) = \frac{U}{U_o} = \frac{1}{1-r^2}, \quad r \neq 1 \quad \text{frequency response function} \quad (4.5) \]

\[ r = \frac{\Omega}{\omega_n} \quad \text{frequency ratio} \quad (4.6) \]

\[ U = U_o H(\Omega) \]
\[ u_p(t) = U_0 H(\Omega) \cos \Omega t \]

\[ p(t) = p_0 \cos \Omega t \]

If \( r < 1 \)

\[ u_p = \left( \frac{U_o}{1 - r^2} \right) \cos \Omega t, \quad \text{in phase} \] \quad (4.9)

If \( r > 1 \)

\[ u_p = \left( \frac{U_o}{r^2 - 1} \right) (- \cos \Omega t) \quad \text{out of phase} \] \quad (4.10)

\[ u(t) = u_p(t) + u_s(t) \]

\[ = \left( \frac{U_o}{1 - r^2} \right) \cos \Omega t + A_1 \cos \omega_s t + A_2 \sin \omega_s t \] \quad (4.11)

\[ A_1, A_2: \quad \text{from } u_o, \dot{u}_o \]

\[ D = \max \left| \frac{u(t)}{U_o} \right| \quad \text{total dynamic magnification factor} \] \quad (4.12)

\[ D_s = \max \left| \frac{u_p(t)}{U_o} \right| = |H(\Omega)| \quad \text{steady-state magnification factor} \] \quad (4.8)
Figure 4.2 Dynamic magnification factors for an undamped SDOF system with \( p(t) = p_0 \sin \omega t \)

**Comments**

- \( D \geq D_s \)

- when \( r = 0 \), \( D = D_s = 1 \): Static response

- when \( r \approx 1 \), \( D \) and \( D_s \) are maximum and very large.

**Example 4.1** steady-state response

\( k = 40 \text{LBS} / \text{in} \)
\( W = 38.6 \text{LBS} \)
\( g = 386 \text{in} / \text{s}^2 \)
\( p_0 = 10 \text{LBS} \)
\( \Omega = 10 \text{ rad/s} \)
\( u(0) = \dot{u}(0) = 0 \)

\[
\dot{u} = \left( \frac{U_o}{1 - r^2} \right) \cos \Omega t + A_1 \cos \omega_n t + A_2 \sin \omega_n t \tag{I}
\]
\[
\dot{u} = -U_0 \Omega \frac{\sin \Omega t - A_i \omega_n \sin \omega_n t + A_z \omega_n \cos \omega_n t}{1 - r^2}
\]  
(2)

\[
\omega_n = \left( \frac{k}{m} \right)^{1/2} = \left( \frac{kg}{W} \right)^{1/2} = \sqrt{\frac{40(386)}{38.6}} = 20 \text{ rad/s}
\]  
(3)

\[
U_0 = \frac{p_o}{k} = \frac{10}{40} = 0.25 \text{ in.}
\]  
(4)

\[
r = \frac{\Omega}{\omega_n} = \frac{10}{20} = 0.5
\]  
(5)

\[
\frac{U_0}{1 - r^2} = \frac{0.25}{1 - (0.5)^2} = \frac{0.25}{0.75} = 0.33 \text{ in.}
\]  
(6)

\[
u(0) = 0 = \frac{U_0}{1 - r^2} + A_i
\]  
(7)

\[
A_i = -\frac{U_0}{1 - r^2} = -0.33 \text{ in.}
\]  
(8)

\[
\dot{u}(0) = 0 = A_z \omega_n
\]  
(9)

\[
A_z = 0
\]  
(10)

\[
u = 0.33[\cos(10t) - \cos(20t)] \text{ in.}
\]  
(11)

**Curves of** \(u_p(t), u_c(t)\) **and** \(u(t)\)
Note

a. The steady-state response has the same frequency as the excitation and is in-phase with the excitation since \( r < 1 \).

b. The forced motion and natural motion alternately reinforce each other and cancel each other giving the appearance of a beat phenomenon. Thus the total response is not simple harmonic motion.

c. The maximum total response \( (u = -0.66 \text{ in. at } t = \pi / 10 \text{s}) \) is greater in magnitude than the maximum steady-state response \( (u_p = 0.33 \text{ in. at } t = 0) \).

- Equation 4.9 and 4.11 are not valid at \( r = 1 \).
- The condition \( r = 1 \), or \( \Omega = \omega_n \), is called resonance, and
- it is obvious from Fig. 4.2 that at excitation frequencies near resonance the response becomes very large.

● When \( r = 1 \),

let

\[ u_p(t) = Ct \sin \Omega t, \quad \Omega = \omega_n \]  \hspace{1cm} (4.13)

then

\[ C = \frac{p_o}{2m \omega_n} \]

\[ = \frac{1}{2} \frac{p_o}{k} \frac{k}{m \omega_n} = \frac{1}{2} U_o \omega_n \]  \hspace{1cm} (4.14)

\[ \therefore u_p(t) = \frac{1}{2} (U_o \omega_n t) \sin \omega_n t \]  \hspace{1cm} (4.15)
§4.2 Response of Viscous-Damped SDOF Systems to Harmonic Excitation

\[ m\ddot{u} + c\dot{u} + ku = p_\omega \cos \Omega t \quad (4.16) \]

Let

\[ u_p = U \cos(\Omega t - \alpha) \quad (4.17) \]

\[ U : \text{ steady-state amplitude} \]
\[ \alpha : \text{ phase} \]

\[ \dot{u}_p = -\Omega U \sin(\Omega t - \alpha) = \Omega U \cos(\Omega t - \alpha + \pi) \quad (4.18) \]
\[ \ddot{u}_p = -\Omega^2 U \cos(\Omega t - \alpha) \]
\[-m\Omega^2 U \cos(\Omega t - \alpha) - c\Omega U \sin(\Omega t - \alpha)\]
\[+ kU \cos(\Omega t - \alpha) = p_o \cos \Omega t\]  \hspace{1cm} (4.19)

When \( m\Omega^2 U < kU \), that is, \( \Omega < \omega_s \).

\[p_o^2 = (kU - m\Omega^2 U)^2 + (c\Omega U)^2\]  \hspace{1cm} (4.20a)

\[\tan \alpha = \frac{c\Omega}{k - m\Omega^2}\]  \hspace{1cm} (4.20b)

Figure 4.5. Force vector polygon

For solution, let

\[u_p(t) = A \sin \Omega t + B \cos \Omega t\]  \hspace{1cm} (4.19a)

\[m\ddot{u} + c\dot{u} + ku = p_o \cos \Omega t\]  \hspace{1cm} (4.16)

(4.19a) \rightarrow (4.16)

\[m\Omega^2 (-A \sin \Omega t - B \cos \Omega t) +\]
\[c\Omega (A \cos \Omega t - B \sin \Omega t) +\]
\[k (A \sin \Omega t + B \cos \Omega t) = p_o \cos \Omega t\]  \hspace{1cm} (4.19b)

\[[(k - m\Omega^2)A - c\Omega B] \sin \Omega t + [(k - m\Omega^2)B + c\Omega A] \cos \Omega t = p_o \cos \Omega t\]

\[(k - m\Omega^2)A - c\Omega B = 0\]  \hspace{1cm} (4.19c)
\[(k - m\Omega^2)B + c\Omega A = p_0 \quad (4.19d)\]

\[(4.19c)\]

\[(k - m\Omega^2)A - c\Omega B = 0 \quad (4.19e)\]

\[B = \frac{k - m\Omega^2}{c\Omega} A \quad (4.19d)\]

\[A = \frac{c\Omega}{(k - m\Omega^2)^2 + (c\Omega)^2} p_0 = \frac{2\xi r}{(1 - r^2)^2 + (2\xi r)^2} U_0\]

\[B = \frac{k - m\Omega^2}{(k - m\Omega^2)^2 + (c\Omega)^2} p_0 = \frac{1 - r^2}{(1 - r^2)^2 + (2\xi r)^2} U_0\]

where

\[U_0 = \frac{p_0}{k} \quad \text{static displacement}\]

\[u_\rho(t) = A\sin \Omega t + B\cos \Omega t \quad (4.19a)\]

\[= \sqrt{A^2 + B^2} \left( \frac{A}{\sqrt{A^2 + B^2}} \sin \Omega t + \frac{B}{\sqrt{A^2 + B^2}} \cos \Omega t \right)\]

\[\sqrt{A^2 + B^2} = \frac{U_0}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}\]

Let

\[\sin \alpha = \frac{A}{\sqrt{A^2 + B^2}} = \frac{2\xi r}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}\]

\[\cos \alpha = \frac{B}{\sqrt{A^2 + B^2}} = \frac{1 - r^2}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}\]

or

\[\tan \alpha = \frac{A}{B} = \frac{2\xi r}{1 - r^2} \quad (4.21b)\]

then
\[ u_\rho(t) = \sqrt{A^2 + B^2} \cos(\Omega t - \alpha) \]

\[ = \frac{U_0}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \cos(\Omega t - \alpha) \]

or

\[ u_\rho(t) = U \cos(\Omega t - \alpha) \]

where

\[ U = \frac{U_0}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \]

amplitude of \( u_\rho(t) \)

\[ D_s = \frac{U}{U_o} = \frac{1}{\left[(1 - r^2)^2 + (2\zeta r)^2\right]^{1/2}} \]

steady-state magnification factor

\[ (4.21a) \]

Figure 4.6. (a) Magnification factor versus frequency ratio for various amounts of damping (linear plot)

Comment
- to compute $\max D_s$, let $\frac{dD}{dr} = 0$

Figure 4.6 (b) Phase angle versus frequency ratio for various amounts of damping (linear plot).

Figure 4.7. (a) Magnification factor versus frequency ratio for various damping factors (logarithmic plot)
Figure 4.7 (b) Phase angle versus frequency ratio for various damping factors (logarithmic frequency scale)

\[ (D_s)_{r=1} = \frac{1}{2\zeta} \]  

(4.22)

The curves of Figs. 4.6 are frequently plotted to logarithmic scales as shown in Figs. 4.7. This is referred to as a Bode plot.

\[
u(t) = \frac{U_o}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}} \cos(\Omega t - \alpha) + e^{-\zeta\omega_d t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)\]

(4.23)

Since the natural motion in Eq. 4.23 dies out with time, it is referred to as a starting transient.

Example 4.2

\[ u(0) = \dot{u}(0) = 0 \]

\[
u = U \cos(\Omega t - \alpha) + e^{-\zeta \omega_d t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) \]  

(1)

\[
U = \frac{U_o}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}} \]  

(2)
\[
\begin{align*}
\omega_n &= \left(\frac{k}{m}\right)^{1/2} = 20 \text{ rad/s} \\
U_o &= \frac{p_o}{k} = \frac{10}{40} = 0.25 \\
r &= \frac{\Omega}{\omega_n} = \frac{10}{20} = 0.5 \\
\zeta \omega_n &= (0.2)(20) = 4 \text{ rad/s}
\end{align*}
\]

\[
U = \frac{0.25}{\sqrt{\left[1 - (0.5)^2\right] + 2(0.2)(0.5)}} = 0.32 \text{ in.}
\]

\[
\tan \alpha = \frac{2\zeta r}{1 - r^2} = \frac{2(0.2)(0.5)}{1 - (0.5)^2} = 0.267
\]

\[
\alpha = 0.26 \text{ rad}
\]

\[
\omega_d = \omega_n \sqrt{1 - \zeta^2} = 20 \sqrt{1 - (0.2)^2} = 19.6 \text{ rad/sec}
\]

\[
\dot{u} = -\Omega U \sin(\Omega t - \alpha) + e^{-\zeta \omega_d t} (A_1 \omega_d - A_2 \zeta \omega_n) \cos \omega_d t - (A_1 \omega_d + A_2 \zeta \omega_n) \sin \omega_d t
\]

\[
u(0) = 0 = 0.32 \cos(-0.26) + A_1
\]

\[
A_1 = -0.32 \cos(-0.26) = -0.31 \text{ in.}
\]

\[
\dot{u}(0) = 0 = -(0.32)(10) \sin(-0.26) + [A_2 (19.6t) + 0.11 \sin(19.6t)] \text{ in.}
\]

\[
A_2 = -0.11 \text{ in.}
\]

\[
u = 0.32 \cos(10t - 0.26) - e^{-2t}[0.31 \cos(19.6t) + 0.11 \sin(19.6t)] \text{ in.}
\]
Other Methods
- Closed Form Solution
  Duhamel Integration Method

- Numerical Methods
  Central difference method
  Average acceleration method
  Linear acceleration method
  Newmark methods
  Wilson method
  Runge Kutta method

Response Spectra